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# MULTIDIMENSIONAL SIGNAL PROCESSING

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The problem of structurally passive synthesis of multidimensional digital filters of the quarter-plane casual type as an interconnection of more elementary building blocks directly in the discrete domain has been addressed via the factorization of the chain matrix, the hybrid matrix and the transfer function matrix associated with a prescribed multi- (over)

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dimensional lossless two-port. By exploiting recent results on the discrete domain representation of such matrices a generalized lossless two-port matrix has been introduced to present all three factorizations in a unified setting. Necessary and sufficient conditions for factorability as well as algorithm for computing these factors when they exist are obtained. In particular, it is shown that in one-dimension the factorizations can always be performed. Thus, in 1-D, discrete domain algorithms for synthesizing previously unpublished internally passive structures as well as alternative methods of synthesis for more conventional structures such as the cascade structure are also obtained as a byproduct of our discussion. Since most multidimensional applications dictate that the filter be either symmetric or (quasi) antimetric, special attention is paid to the problem of synthesis of these subclasses of multidimensional lossless two-ports.

Due to difficulties inherent to the mathematics of quarter plane multidimensional filtering, (e.g., polynomial nonfactorability etc.) the alternative approach of studying passive filters of the fully recursive half-plan casual type is undertaken in chapter 3 for the first time in the literature. Apart from the tractability of analysis and design, such filters have the potential to maximally exploit the currently emerging parallel (VLSI and/or optical architecture, when implementation is called for. Thus, passive and lossless two-dimensional digital one-ports as well as two-ports of the fully recursive half-plane type are introduced and are characterized in terms of their transfer function descriptions. An algorithm for the structurally passive synthesis of filters having such recursive structure is then derived from these representation results as an extension of a recent Schur type algorithm for the synthesis of discrete lossless two-ports. Design methods for important practical cases when frequency response of the filter is required to have specific symmetries are also planned.

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## INDEX

# CHAPTER 1
## INTRODUCTION

The need for multidimensional signal processing in manifold areas of applications is now well recognized. The important role of passive digital filtering and related multidimensional modelling schemes in the general area of multidimensional signal processing, and in particular, in the areas of image processing, target tracking etc., has already been documented in an earlier report [1] and thus, its discussion will not be undertaken here. A brief description of the results obtained during the course of the present investigation subsequent to those reported in [1] follows.

In chapter 1 properties of various multidimensional polynomials arising in studies of passive (lossless) discrete multidimensional systems are investigated. Reactance Schur polynomials and immittance Schur polynomials occurring respectively as the denominators (and numerators) of discrete reactance functions and discrete positive functions are introduced and their properties studied. Role of these polynomials in scattering or immittance descriptions of passive discrete time domain multiports are brought out. The interrelation between classes of multidimensional polynomials arising in discrete systems and the corresponding classes of polynomials in the context of continuous systems is also studied via the mechanics of bilinear transformation.

In chapter 2 the problem of structurally passive synthesis of multidimensional digital filters of the quarter-plane causal type as an interconnection of more elementary building blocks directly in the discrete domain has been addressed via the factorization of the chain matrix, the hybrid matrix and the transfer function matrix associated with a prescribed multidimensional lossless two-port. By exploiting recent results on the discrete domain representation of such matrices a generalized lossless two-port matrix has been introduced to present all three factorizations in an unified setting. Necessary and sufficient conditions for factorability as well as algorithm for computing these factors when they exist are obtained. In particular, it is shown that in one-dimension the factorizations can always be performed. Thus, in 1-D, discrete domain algorithms for synthesizing previously unpublished internally

passive structures as well as alternative methods of synthesis for more conventional structures such as the cascade structure are also obtained as a byproduct of our discussion. Since most multidimensional applications dictate that the filter be either symmetric or (quasi) antimetric, special attention is paid to the problem of synthesis of these subclasses of multidimensional lossless two-ports.

Due to difficulties inherent to the mathematics of quarter plane multidimensional filtering, (e.g., polynomial nonfactorability etc.) the alternative approach of studying passive filters of the fully recursive half-plane causal type is undertaken in chapter 3 for the first time in the literature. Apart from the tractability of analylis and design, such filters have the potential to maximally exploit the currently emerging parallel (VLSI and/or optical) architechture, when implementation is called for. Thus, passive and lossless two-dimensional digital one-ports as well as two-ports of the fully recursive half-plane type are introduced and are characterized in terms of their transfer function descriptions. An algorithm for the structurally passive synthesis of filters having such recursive structure is then derived from these representation results as an extension of a recent Schur type algorithm for the synthesis of discrete lossless two-ports. Design methods for important practical cases when frequency response of the filter is required to have specific symmetries are also presented.

Finally, results are summarized, conclusions are drawn and recommendations for further work are made in chapter 4.

Each of the following chapters are self contained and can be read independently. For similar discussions in the open literature we refer to the publications [2], [3] and [4] in the following.

## References


[1] S. Basu, Multidimensional Filtering Investigations, Technical report, RADC-TR-86-231. A178 745.

[2] S. Basu, A. Fettweis, New results on stable multidimensional polynomials, Part II: Discrete case, IEEE trans. on CAS, November, 1987.

[3] S. Basu and A. Tan, On the synthesizability of multidimensional lossless two-ports, Midwest Symp. on Circuits and Systems, Syracuse, 1987. Also, submitted for journal publication.

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# CHAPTER 2
# SOME NEW RESULTS ON STABLE MULTIDIMENSIONAL DIGITAL FILTERS

## 2.1. INTRODUCTION:

The scattering Hurwitz polynomials have been introduced recently as the denominators of (rational) bounded functions arising in studies on passive multidimensional (k-D) networks [1]. Subsequently, the denominator polynomials of (rational) reactance functions and (rational) positive functions have also been characterized as the reactance Hurwitz and the immittance Hurwitz polynomials respectively [2]. We also refer to [3] for a related discussion. In view of recent interest in synthesis of passive digital filter networks directly in the discrete domain, a study of properties of the corresponding polynomials arising in discrete time systems seems highly relevant. Furthermore, it is now known to researchers in the field that unlike the 1-D case, in multidimensions discrete counterparts of certain continuous domain results need not always be true [4-6]. A separate treatment for discrete time systems is thus needed for a more careful analysis. The first step in this direction has been taken in [7] by studying the properties of discrete scattering Hurwitz polynomials. In the present paper further properties of this latter class of polynomilas are studied, discrete reactance Hurwitz polynomials and discrete immittance Hurwitz polynomials, ocurring as the denominators (and consequently the numerators) of discrete reactance functions and discrete positive functions, are introduced and their properties, as they relate to both single-port and multi-port passive digital networks, are studied in detail. It is possible, at least in principle, to undertake the abovementioned discussion via the utilization of multiple bilinear transform and analogous results already existing [2] for continuous systems. However, it turns out, due to difficulties of the type elaborated in [4-6], that such an indirect approach is neither mathematically elegant nor is it desirable from the standpoint of a self-contained theory of passive multidimensional discrete systems. Therefore, the present paper has been organized in such a way that it can be read indepedently of all existing publications on continuous systems.

In what follows we will consider polynomials $a=a(\underline{z})$ in k-variables $\underline{z} = (z_1,z_2,\ldots z_k)$. All polynomials and rational functions are assumed to be functions of k-variables unless otherwise specified. A k-variable polynomial a will said to <u>involve</u> a variable $z_i$ if the indeterminate $z_i$ actually exists in

at least one of the monomials comprising the polynomial a. A polynomial will be called non-trivial if it involves at least one of the variables $z_i$, i = 1 to k. More generally, the above comments apply if a is a rational matrix function in $\underline{z} = (z_1, z_2, \ldots z_k)$. If $a = a(\underline{z})$ is written as a polnomial in $z_i$ as:

$$a = \sum_{\nu=0}^{n_i} a_\nu z_i^\nu \qquad (2.1.1)$$

where the coefficients $a_\nu$'s are polynomials in the remaining variables with $a_{n_i} \neq 0$, then $n_i$ is called the partial degree of a in the variable $z_i$ and is to be denoted by $\deg_i a$. Occassionally, we will write in a compact notation $\underline{z}' = (z_2, z_3, \ldots z_k)$. Two polynomials will be said to be relatively prime if they do not have a proper (i.e., a non-constant) factor in common. A rational function will be said to be in irreducible form if the numerator and denominator polynomials are relatively prime. The discrete paraconjugate $\hat{a}$ of a polynomial $a = a(z_1, z_2, \ldots, z_k)$ in k-variables $\underline{z} = (z_1, z_2 \ldots, z_k)$ is defined as:

$\hat{a} = \tilde{a}.\underline{z}^{\underline{n}_a}$, where $\tilde{a} = a_*(z_1^{*-1}, z_2^{*-1}, \ldots, z_k^{*-1})$, and the notation $\underline{z}^{\underline{n}_a}$ denotes the monomial $z_1^{n_1} z_2^{n_2} \ldots z_k^{n_k}$; $n_i = \deg_i a$. The superscript * denotes complex conjugation. Note that the discrete paraconjugate of $\hat{a}$ is not necessarily equal to a. The polynomial a will be called discrete selfparaconjugate if $\hat{a} = \gamma a$, where $\gamma$ is a (necessarily unimodular) constant. In addition, a will be called discrete paraeven or discrete paraodd according as $\gamma = 1$ or $\gamma = -1$. The notation $|\underline{z}| < \underline{1}$ will mean $|z_i| < 1$ for all i = 1 to k. Obvious variations of this notation with the symbol > replaced by =, <, $\leq$, $\geq$ etc. will be used.

The terms sequentially almost complete set $\Theta$ or sequentially infinite set $\Theta$, as defined in [2] will be used. In the present context, however, the elements of the m-tuple $(\theta_1, \theta_2, \ldots \theta_m)$ belonging to $\Theta$ will, unless otherwise specified, be chosen from the field of real numbers modulo $2\pi$.

We will have the occasion to consider the polynomial $a_z$ associated with the polynomial a, as defined in the following. Let $a = a(z) = a_0 + a_1 z + \ldots a_n z^n$, where the coefficients $a_i$'s are polynomials in several variables over the field of complex numbers. Then associated with a we define a polynomial $a_z$ via the formal algebraic operation as: $a_z = a_z(\underline{z}) = b_0 + b_1 z + \ldots b_n z^n$, where $b_i = (n-2i)a_i$ for $i = 0, 1, \ldots n$. Further elementary properties of the polynomial $a_z$ are derived

in appendix 2.C. More important definitions will be introduced as they become necesary in the main text.

Various classes of polynomials devoid of zeros in the unit polydisc will, in general, be referred to as the multidimensional Schur polynomials. This is a departure from the previously adopted terminology in [7],[8], but is consistent with the terminology used for the corresponding class of one-variable polynomials in the literature. Properties of the widest sense Schur polynomials, selfparaconjugate Schur polynomials and scattering Schur polynomials are discussed in section 2.2. Section 2.3 contains discussion on elementary properties of multidimensional discrete positive functions. In section 2.4 reactance Schur and immittance Schur polynomials respectively occuring as the numerators (and consequently denominators) of discrete reactance functions and discrete positive functions are introduced. Extensions of some of these results to passive multiports are discussed in section 2.5. Finally, interrelationship between the various classes of multidimensional Schur polynomilas and the corresponding classes of multidimensional Hurwitz polynomials is examined via the artifice of bilinear transformation in section 2.6, and conclusions are drawn in section 2.7. An appendix is included, where some general properties of multidimensional polynomials proved elsewhere are stated, and a few basic formulae occurring in studies of discrete systems are derived. For consistency in the logical sequence in which the proofs of results are arranged in the present paper, theorems 2.2.2.7 and 2.2.2.8 should be read after 2.3.6 and before corollary 2.2.3.7. However, theorems 2.2.2.7 and 2.2.2.8 are incorporated earlier in section 2.2.2 rather than in section 2.2.3, beacuse this allows for a more systematic categorization of the properties of various classes of multidimensional Schur polynomials.

## 2.2. VARIOUS CLASSES OF SCHUR POLYNOMIALS

### 2.2.1. Widest sense Schur polynomials:

Definition 2.2.1.1: A polynomial a is called widest sense Schur if $a \neq 0$ for $|\underline{z}| < \underline{1}$.

Theorem 2.2.1.2: If $a(\underline{z})$ is a widest sense Schur polynomial in k-variables $\underline{z}=(z_1, z_2, \ldots z_k)$, then the polynomial $a'(\underline{z}')$ in (k-1) variables $\underline{z}'$ obtained by freezing $z_1$ in $|z_1| \leq 1$ is either widest sense Schur or is identically equal to zero. Furthermore, the latter instance can occur only for finitely many values of $z_1$ on $|z_1| = 1$.

Proof: Obviously $a'(\underline{z}')$ cannot be zero in $|\underline{z}'|<1$ for any $|z_1|<1$, because if for some $z_1 = z_{10}$ in $|z_{10}|<1$, $a'(\underline{z}')$ has a zero $\underline{z}' = \underline{z}'_0$ with $|\underline{z}'_0| < \underline{1}$, then $a(z_{10}, \underline{z}'_0) = 0$, which contradicts the widest sense Schur property of $a(\underline{z})$. Furthermore, if $a'(\underline{z}') = 0$ for some $\underline{z}' = \underline{z}'_0$ in $|\underline{z}'_0|<1$, and for $z_1 = z_{10}$ on $|z_1|=1$ then the following two possibilities arise. Firstly, if $a'(\underline{z}') \equiv 0$ i.e., $a(z_{10}, \underline{z}') \equiv 0$ for any $\underline{z}'$ then $(z-z_{10})$ is a factor of $a(\underline{z})$. Obviously, then there are finitely many values of $z_{10}$ on $|z_1|=1$, such that $(z_1-z_{10})$ is a factor of $a(\underline{z})$. This proves the second part of assertion of the theorem. Next, to prove the first part, assume that $a'(\underline{z}') \not\equiv 0$, i.e., $(z_1-z_{10})$ is not a factor of $a(\underline{z})$. Then by moving the variable $z_1$ by sufficiently small amount inside $|z_1|<1$ from $z_1=z_{10}$, and by invoking the continuity property of zeros of a polynomial we would be able to construct a zero of $a(\underline{z})$ with $|z_1|<1$ and $|\underline{z}'_0| < \underline{1}$, which contradicts the widest sense Schur property of $a(\underline{z})$ and is thus impossible.

A repeated application of the above theorem, along with the definition of sequentially almost complete [cf. Appendix A] set yield the following result.

Theorem 2.2.1.3: let $a(\underline{z})$ be a widest sense Schur polynomial in k-variazles $\underline{z}$. Then there exists a sequentially almost complete set of m-tuples $\Theta_m$, of order m<k, such that the (k-m) variable polynomial obtained by freezing m of the k variables $z_i$ at $z_i = z_{i0}$ with $z_i = \exp(j\theta_i)$, $0 \leq \theta_i < 2\pi$ for, say i = 1 to m,

is widest sense Schur if $(\theta_1, \theta_2, \ldots \theta_m) \varepsilon \Theta_m$ and is identically equal to zero if $(\theta_1, \theta_2, \ldots \theta_m) \not\varepsilon \Theta_m$.

### 2.2.2 Selfparaconjugate Schur Polynomials:

<u>Definition 2.2.2.1:</u> A polynomial a is called self-paraconjugate Schur if $a \neq 0$ for $|\underline{z}| \leq \underline{1}$ and $\hat{a} = \gamma a$, where $\gamma$ is a complex number.

Since $\hat{a} = \tilde{a} \prod_{i=1}^{k} z_i^{n_i}$ we have for $z_i = \exp(j\theta_i)$, i=1 to k, $|\hat{a}| = |\tilde{a}| = |a^*| = |a|$. Thus the constant $\gamma$ in definition 2.2.2.1 is necessarily unimodular i.e., $|\gamma| = 1$.

<u>Lemma 2.2.2.2:</u> Let a be a selfparaconjugate Schur polynomial in k-variables $\underline{z}$, and $a_\lambda \not\equiv 0$ be the (k-1) variable polynomial obtained by freezing in a one of the variables, say $z_\lambda$, on $|z_\lambda| = 1$. Then $\deg_i a = \deg_i a_\lambda$ for the remaining variables i = 1 to k but $i \neq \lambda$.

<u>Proof:</u> Assume $\lambda \neq 1$. Let us write the polynomial a as in (2.1.1) with i=1. Then it follows from property 2.B8 in the appendix that $\gamma a_{n_1} = \tilde{a}_0 \prod_{i=2}^{k} z_i^{n_i}$. Furthermore, since $\deg_i a_0 \leq n_i$, it follows from the last equation that $\gamma a_{n_1} = \hat{a}_0 ( \prod_{i=2}^{k} z_i^{p_i})$, where $p_i$ are some nonnegative integers. If $a_{n_1} \equiv 0$ for some $z_\lambda = z_{\lambda 0}$ on $|z_\lambda| = 1$ irrespective of the other variables then $(z - z_{\lambda 0})$ must be a factor of $a_{n_1}$, and thus a factor of $\hat{a}_0$. This later conclusion, however imply that $(z_\lambda - z_{\lambda 0})$ is a factor of $a_0$, and therefore $a=0$ for $z_\lambda = z_{\lambda 0}$ and $z_1 = 0$ irrespective of other variables, and consequently, either $a_\lambda \equiv 0$ or $a_\lambda$ is not widest sense Schur. However, since $a_\lambda \not\equiv 0$ by hypothesis, this is in contradiction with theorem 2.1.1. Therefore, $a_{n_1} \not\equiv 0$ for any $z_\lambda$ on $|z_\lambda| = 1$. The validity of the present lemma for other values of i can similarly be demonstated by writting a as a polynomial in $z_i$ with coefficients as polynomials in remaining variables.

<u>Theorem 2.2.2.3:</u> If a is a self-paraconjugate Schur polynomial in k-variables then the polynomial a' in (k-1) variables obtained by freezing any one of the variables, say $z_1$, is also self-paraconjugate Schur for an almost complete set of values on $|z_1| = 1$

<u>Proof:</u> From theorem 2.1.1, a' is widest sense Schur for an almost complete set

of values of $\theta_1$ in $0\leq\theta_1<2\pi$, where $z_1=\exp(j\theta_i)$ i.e., for an almost complete set $z_1$ on $|z_1| = 1$. It remains to show that $a' = \gamma'.a'$, where $\gamma'$is a constant. By substituting $z_1=\exp(j\theta_1)$ in $\hat{a}=\gamma a$ it follows that $a^*(\exp(j\theta_1), z_2^{*-1}, \ldots z_k^{*-1})(\exp(j\theta_1 n_1)) \prod_{i=2}^{k} z_i^{n_i} = \gamma a((\exp(j\theta_1), z_1, \ldots z_k)$, where $n_i=\deg_i a$. The desired result then follows by noting that the right hand side of the last equality is $\gamma a'$, whereas the left hand side due to the fact that $\deg_i a=\deg_i a'=n_i$ (lemma 2.2.2.2)is equal to $(\exp(jn_1\theta_1))\hat{a}'$.

Repeated use of the above result yields the following theorem.

<u>Theorem 2.2.2.4:</u> Let a be a self-paraconjugate Schur polynomial and a be the polynomial obtained by freezing m, m<k, of the k-variables, say $z_i$, i = 1 to m, on $|z_i| = 1$ i.e., $z_i = \exp(j\theta_i)$ ; $0\leq\theta_i<2\pi$. Then there exists a sequentially almost complete set $\Theta_m$ of m-tuples of order m such that for any $(\theta_1, \theta_2, \ldots \theta_m)\varepsilon\Theta_m$, the polynomial $a_m$ is selfparaconjugate Schur.

<u>Theorem 2.2.2.5:</u> Let a be a self-paraconjugate Schur polynomial. There exists a sequentially almost complete set $\Theta$ of k-tuples of order (k-1) such that a = 0 for any $z_i = \exp(j\theta_i)$; $0\leq\theta_i<2\pi$, i = 1 to k, and $(\theta_1, \theta_2, \ldots \theta_k)\varepsilon\Theta$.

<u>Proof:</u> From theorem 2.2.2.4, the polynomial $a_{k-1}$ obtained by freezing (k-1) of the k variables, say $z_i$, i=1 to (k-1) on $z_i = \exp(j\theta_i)$ ; $0\leq\theta_i<2\pi$ is selfparaconjugate Schur for any $(\theta_1, \theta_2, \ldots \theta_{k-1})\varepsilon\Theta_{k-1}$, where $\Theta_{k-1}$ is a sequentially almost complete set of order (k-1). Since the polynomial $a_{k-1} = a_{k-1}(z_k)$ is self-paraconjugate Schur, $a_{k-1}\neq 0$ for $|z_k|<1$ as well as for $|z_k|>1$, and thus $a_k$ has zeros on $|z_k|=1$ only. Therefore, there is a sequentially almost complete set $\Theta$ of k-tuples of order (k-1) such that a = 0 for $(\theta_1, \theta_2, \ldots \theta_k)\varepsilon\Theta$.

We refer ahead to definition 2.2.3.1 for the statement, but not the proof of the following theorem.

<u>Theorem 2.2.2.6:</u> A widest sense Schur polynomial a can be expressed as a product of a selfparaconjugate Schur factor and a scattering Schur factor.

<u>Proof:</u> Let a=de, $\hat{a}$=df, where d is the gcd between a and $\hat{a}$, and thus e and f are prime polynimials. From property 2.B6 in the appendix then $\hat{d} = \gamma d$. i.e.,

d is discrete self-paraconjugate. Since a is widest sense Schur so are both d and e. Thus d is selfparaconjugate Schur. We further claim that e is scattering Schur. Clearly, $d.f = \hat{a} = \hat{d}.\hat{e} = \gamma.d.\hat{e}$ and hence $f = \gamma\hat{e}$, where the second equality follows from property 2.B1 in the appendix. Therefore, if e and $\hat{e}$ had a common factor then e and f would not be relatively prime.

The following theorem characterizes selfparaconjugate Schur polynomials.

<u>Theorem 2.2.2.7:</u> A polynomial a is self-paraconjugate Schur if and only if $a \neq 0$ for $|\underline{z}|<\underline{1}$ as well as for $|\underline{z}|>\underline{1}$.

<u>Proof:</u> Necessity: $a \neq 0$ in $|\underline{z}|<\underline{1}$ by definition. Furthermore, since a is discrete self-paraconjugate, a zero of a in $|\underline{z}|>\underline{1}$ would imply existence of a zero of a in $|\underline{z}|<\underline{1}$, and is hence excluded.

Sufficiency: If $a \neq 0$ for $|\underline{z}|<\underline{1}$, then by theorem 2.2.2.6 a is product of a self-paraconjugate Schur factor and a scattering Schur factor. However, a factor of latter type, due to theorem 2.2.3.6 must have a zero in $|\underline{z}|>\underline{1}$, thus implying a zero of a in $|\underline{z}|>\underline{1}$, which is excluded.

<u>Theorem 2.2.2.8:</u> The irreducible factors of a selfparaconjugate polynomial a are also selfparaconjugate Schur.

<u>Proof:</u> Obviously, the irreducible factors of a are widest sense Schur. Thus, due to theorem 2.2.2.6, these factors are either selfparaconjugate Schur or scattering Schur. However, presence of a factor of the latter type, due to theorem 2.2.3.6, would indicate that a has a zero in $|\underline{z}|>\underline{1}$, which is ruled out in theorem 2.2.2.7.

<u>Lemma 2.2.2.9:</u> Let a be a selfparaconjugate Schur polynomial involving the variable $z_i$. Then $\deg_\lambda a = \deg_\lambda a_{z_i}$ for $\lambda = 1$ to k.

<u>Proof:</u> If $\lambda=i$ the proof is obvious from the definition of $a_{z_i}$. Next, let $\lambda\neq i$, and a be written as in (2.1.1). Since a is selfparaconjugate Schur, and in particular widest sence Schur, $a_0$ cannot have a factor $z_\lambda$ for any $\lambda$, beacuse otherwise a would be zero for $z_\lambda=0$, $z_i=0$, and arbitrary values of the remaining variables. Therefore, $a_0$ must contain a monomial not involving $z_\lambda^{-1}$. Since from

property 2.B8 in the appendix, $a_{n_i} = \gamma\tilde{a}_0 \prod_{\ell \neq i} z_\ell^{n_\ell}$, the last stated property of $\tilde{a}_0$ yields that $\deg_\lambda a_{n_i} = n_\lambda$. Furthermore, since in $a_{z_i} = \sum_{\nu=0}^{n_i} b_\nu z_i^\nu$, $b_{n_i} = -n_i . a_{n_i}$ we have $\deg_\lambda b_{z_i} = \deg_\lambda a_{n_i} = n_\lambda$. Consequently, $\deg_\lambda a_{z_i} = n_\lambda$.

### 2.2.3 Scattering Schur Polynomials:

Definition 2.2.3.1: A polynomial a is called scattering Schur if $a \neq 0$ for $|\underline{z}| < \underline{1}$, (i.e., a is widest sense Schur) and if a and $\hat{a}$ do not have any common factor.

The term discrete scattering Hurwitz has earlier been used for the above class of polynomials in [7],[8].

Theorem 2.2.3.2: If a is a scattering Schur polynomial, then a cannot have a discrete selfparaconjugate factor.

Proof: If a had a factor d i.e., a = de, with $\hat{d} = \alpha d$, where $\alpha$ is a constant, then $a = \hat{d}\hat{e} = d\alpha\hat{e}$, and thus a and $\hat{a}$ would not be relatively prime.

Corollary 2.2.3.3: A scattering Schur polynomial $a = a(\underline{z})$ in one variable is a strict sense Schur polynomial i.e., $a \neq 0$ for $|z| \leq 1$.

Theorem 2.2.3.4: Let a be a scattering Schur polynomial in k-variables then the polynomial a′ in (k-1) variables obtained by freezing any one of the variables, say $z_1$, with $z_1 = \exp(j\theta_1)$; $0 \leq \theta_1 < 2\pi$, is also scattering Schur for an almost complete set of values of $\theta_1$.

Proof: Let $a'(\underline{z}') = a(z_{10}, \underline{z}')$ with $z_{10} = \exp(j\theta_{10})$, where $\theta_{10}$ is a fixed value of $\theta_1$ in $0 \leq \theta_1 < 2\pi$. Due to theorem 2.2.1.2, a′ is either identically equal to zero or is a widest sense Schur polynomial. However, in the former case $(z - z_{10})$, which is selfparaconjugate Schur, is a factor of a, and thus excluded due to theorem 2.2.3.2.

Next, the polynomials a and $\hat{a}$ are relatively prime by definition 2.2.3.1, and thus the (k-1) variable polynomials $a' = a(\underline{z}') = [a]_{z_1 = z_{10}}$ and $[\hat{a}]_{z_1 = z_{10}}$ with $z_{10} = \exp(j\theta_{10})$, due to property 2.A4 in the appendix, are relatively prime for an almost complete set $\Theta$ of values of $\theta_{10}$. However, since from property

2.B7, $[\hat{a}]_{z_1=z_{10}} = z_{10}^{n_1}\hat{a}' \prod_{i=2}^{k} z_i^{p_i}$, $p_i \geq 0$, we have that a' and $\hat{a}'$ are relatively prime polynomials, and thus a' is scattering Schur for all $\theta_{10} \varepsilon \Theta$.

Extending the above result via a repeated application we obtain the following:

Theorem 2.2.3.5: Let a be a scattering Schur polynomial and let $a_m$ be the polynomial obtained from a by freezing m, m<k of the k-variables, say $z_i$, i = 1 to m on $z_i = \exp(j\theta_i)$, $0 \leq \theta_i < 2\pi$. Then $a_m$ is also scattering Schur for any choice of the m-tuple $(\theta_1, \theta_2, \ldots \theta_m) \varepsilon \Theta_m$, where $\Theta_m$ is a sequentially almost complete set of m-tuples of order m.

Theorem 2.2.3.6: A scattering Schur polynomial a must have zeros in $|\underline{z}| > \underline{1}$.

Proof: Let $a_{k-1}$ be the polynomial obtained from a by freezing (k-1) of the k-variables, say $z_i$, for i = 1 to (k-1) on $z_i = \exp(j\theta_i)$ $0 \leq \theta_i < 2\pi$. Then due to theorem 2.2.3.5, there exists a sequentially almost complete set $\Theta_{k-1}$ of order (k-1) such that for any $(\theta_1, \theta_2, \ldots \theta_{k-1}) \varepsilon \Theta_{k-1}$ the polynomial $a_{k-1}$ is scattering Schur, and is thus, in view of corollary 2.2.3.3, a strict sense Schur polynomial in the variable $z_k$ only. Therefore, $a_{k-1}$ has zeros in $|z_k| > 1$. Consequently, a has zeros for $|z_i| = 1$, i = 1 to (k-1) and $|z_k| > 1$. Next, by continuously moving the variables $z_i$ in the regions $|z_i| > 1$ for i = 1 to (k-1) by sufficiently small amounts, and invoking the continuity property of zeros of a polynomial as a function of its coefficients, it follows that a has zeros in $|\underline{z}| > \underline{1}$.

Theorem 2.2.3.6 can, in fact, be strengthened in the following form.

Corollary 2.2.3.7: An irreducible polynomial a is scattering Schur if and only if $a \neq 0$ for $|\underline{z}| < \underline{1}$ and a has at least one zero in $|\underline{z}| > \underline{1}$.

Proof: Necessity of the theorem is obvious in view of theorem 2.2.3.6. To prove sufficiency, let us note that a is widest sense Schur, and therefore, by virtue of its irreducibility and by theorem 2.2.2.6 is either a selfparaconjugate Schur or a scattering Schur polynomial. However, due to theorem 2.2.2.7, a self-paraconjugate Schur polynomial cannot have zeros in $|\underline{z}| > \underline{1}$. Thus, a is a scattering Schur polynomial.

Theorem 2.2.3.8: Let a be a widest sense Schur polynomial. Then a is also scattering Schur if and only if the zeros of a on $|\underline{z}|=\underline{1}$ does not form a sequentially almost complete set of order (k-1).

Proof: Sufficiency: Assume for contradiction that d is the nonconstant greatest common factor between a and $\hat{a}$. By property 2.B6 in the appendix d is a discrete self-paraconjugate polynomial. Invoking theorem 2.2.2.5, it follows that a = 0 for $z_i=\exp(j\theta_i)$, $0\leq\theta_i<2\pi$, for i = 1 to k with $(\theta_1,\theta_2,\ldots\theta_k)\varepsilon\Theta$, where $\Theta$ is a sequentially almost complete set of order (k-1). However, this latter conclusion contradicts the fact that the zeros of a on $|\underline{z}|=\underline{1}$ cannot form a sequentially almost complete set of order (k-1). The polynomial d is therefore a constant, thus proving the scattering Schur property of a.

Necessity: If a=0 for $z_i = \exp(j\theta_i)$, $0\leq\theta_i<2\pi$, i = 1 to k with $(\theta_1,\theta_2,\ldots\theta_k)\varepsilon\Theta$, where $\Theta$ is a sequentially infinite set of order (k-1), then $a(\exp(j\theta_1), \exp(j\theta_2),\ldots \exp(j\theta_k)) = \hat{a}^*(\exp(j\theta_1),\ldots\exp(j\theta_k)) \prod_{i=1}^{k} \exp(jn_i\theta_i)=0$, where $n_i = \deg_i a$, thus implying that a and $\hat{a}$ would have sequentially infinitely many common zeros of order (k-1) on $|\underline{z}| = \underline{1}$. Property 2.A3 in the appendix then implies that a and $\hat{a}$ would have a common factor, which is impossible if a is scattering Schur.

Theorem 2.2.3.9: (a) Factors of scattering Schur polynomials are scattering Schur. (b) Conversely, products of scattering Schur polynomials are also scattering Schur.

Proof: (a) Let a=bc be a scattering Schur polynomial. Obviously, then b and c are widest sense Schur, beacuse a is so. Furthermore, since due to property 2.B1 in the appenix, $\hat{a}=\hat{b}\hat{c}$, there cannot be a nontrivial common factor between b and $\hat{b}$ or between c and $\hat{c}$, beacuse otherwiswe a and $\hat{a}$ would not be relatively prime, and the scattering Schur property of a would thus be violated. Therefore, b and c are both scattering Schur polynomias.

(b) Conversely, If a=bc with b and c scattering Schur, then clearly a is widest sense Schur. Due to theorem 2.2.2.6 a is the product of a scattering Schur factor and a selfparaconjugate Schur factor. However, irreducible factors of the latter type are also selfparaconjugete Schur due to theorem 2.2.2.8, and

would thus be contained in either b or c, which in view of theorem 2.2.3.2, violates the scattering Schur property of b or c. Therefore, a is a Scattering Schur polynomial.

<u>Theorem 2.2.3.10:</u> Let a be a scattering Schur polynomial in k variables, then the (k-1) variable polynomial a′ obtained by freezing any one of the variables, say $z_i$, in $|z_i|<1$ is also scattering Schur.

<u>Proof:</u> Assume i = 1, and $z_1$ to be frozen at $z_1 = z_{10}$. Obviously, $a' = a'(\underline{z}') = a(z_{10},\underline{z}')$ is widest sense Schur. We only need to show that a′ and $\hat{a}'$ are relatively prime polynomials. If a′ and $\hat{a}'$ are not relatively prime then the greatest common factor d between them is by property 2.B6 discrete self-paraconjugate, and thus is a selfparaconjugate Schur polynomial. Invoking theorem 2.2.2.5 it then follows that d=0, and thus a = 0, for $z_i = \exp(j\theta_i)$; $0 \leq \theta_i<2\pi$ for i = 1 to k, with $\underline{\theta}' = (\theta_2,\theta_3,\ldots\theta_k)\varepsilon\Theta'$, where $\Theta'$ is a sequentially almost complete set of order (k-2). Since a = 0 for $z_i = \exp(j\theta_i)$ $0\leq\theta_i<2\pi$, i = 2 to k with $\underline{\theta}'\varepsilon\ \Theta$ and for $z_1 = z_{10}$, theorem 2.2.1.3 along with the fact $|z_{10}|<1$ yields that a = 0 for $z_i = \exp(j\theta_i)$, $0\leq\theta_i<2\pi$, with $\underline{\theta}'\varepsilon\ \Theta'$ and arbitrary $z_1$. Therefore, by restricting $z_1$ on $z_1 = \exp(j\theta_1)$ it is possible to assert that a = 0 for $z_i = \exp(j\theta_i)$, $0\leq\theta_i<2\pi$, for all $\underline{\theta}=(\theta_1,\theta_2,\ldots\theta_k)\ \varepsilon\ \Theta$, where $\Theta$ is a sequentially almost complete set, and thus a sequentially infinite set of k-tuples of order (k-1). This latter conclusion, in view of theorem 2.2.3.8 violates the scattering Schur property of a. The polynomials a and a′ are thus relatively prime.

<u>Lemma 2.2.3.11:</u> If a is a widest sense Schur polynomial and b a further polynomial such that $|b/a|\leq1$ for $\underline{z}$ on $|\underline{z}|=1$ whenever $a\neq0$, then $|b/a|\leq1$ for $|\underline{z}|<\underline{1}$. Furthermore, if $|b/a|=1$ for some $\underline{z}$ in $|\underline{z}|=\underline{1}$ then it is impossible to have $|b/a|=1$ for some $\underline{z}$ in $|\underline{z}|<\underline{1}$ unless b/a is independent of $z_i$ for all i i.e., a constant.

<u>Proof:</u> Assume for the purpose of induction that the theorem is true for polynomial and rational functions in (k-1) variables. Then due to theorem 2.2.2.3, the polynomial a′ in (k-1) variables obtained by freezing $z_1$ in a at $z_{10}$ on $|\underline{z}|=\underline{1}$ is widest sense Schur for almost all (i.e., except finitely many) choices of $z_{10}$ on $|\underline{z}|=\underline{1}$. Also, let us define the (k-1) variable polynomial

$d'=d'(\underline{z}')=d(z_{10},\underline{z})$. Now since $a'=a'(\underline{z}')\neq 0$ implies $a(z_{10},\underline{z}')\neq 0$, if $|b/a|\leq 1$ for $|\underline{z}|=\underline{1}$, whenever $a\neq 0$ then we have that $|b'/a'|\leq 1$ for $|\underline{z}'|=\underline{1}$, whenever $a'\neq 0$. Consequently, by invoking induction hypothesis we may assert that $|b'/a'|\leq 1$ for $|\underline{z}'|<\underline{1}$.

Consider next the polynomial $a_1=a(z_1,\underline{z}_0')$ and $b_1=b(z_1,\underline{z}_0')$, where $\underline{z}_0'$ is considered frozen in $|\underline{z}'|<\underline{1}$. Obviously, $a_1\neq 0$ in $|z_1|<1$. Furthermore, from the conclusion of the last paragraph it follows that $|a_1/b_1|\leq 1$ for almost all $z_1$ on $|z_1|=1$. Thus, via an application of the known maximum modulus principle for rational functions of one variable, we conclude that $|b_1/a_1|\leq 1$ for $|z_1|<\underline{1}$. Since $\underline{z}_0'$ is chosen arbitrarily in $|\underline{z}'|<\underline{1}$, the latter conclusion yields $|b/a|\leq 1$ for $|\underline{z}|<\underline{1}$.

Finally, the last statement of the present theorem follows from the the known result [9] that a function of several complex variables cannot reach a maximum at a point interior to the domain of holomorphy unless it is a constant.

<u>Theorem 2.2.3.12:</u> (a) If a is a nontrivial scattering Schur polynomial then $|\tilde{a}/a|\leq 1$ for $|\underline{z}|<\underline{1}$. (b) Conversely, if b and a are relatively prime polynomials such that $|b/a|\leq 1$ for $|\underline{z}|<\underline{1}$ then a is either a scattering Schur polynomial or a nonzero constant.

<u>Proof:</u> (a) Since $|\hat{a}/a|=1$ for $|\underline{z}|=\underline{1}$, whenever $a\neq 0$, the result follows immediately from lemma 2.2.3.11.

(b) Clearly, $a\neq 0$ for $|\underline{z}|<\underline{1}$, beacuse if $a(\underline{z}_0)=0$ with $|\underline{z}_0|<\underline{1}$ then in order for $|b/a|<1$ to be satisfied we would necessarily have $b(\underline{z}_0)=1$ i.e., the rational function b/a would have a nonessential singularity of second kind at $\underline{z}_0$. Consequently, b/a would also have a singularity of first kind in an arbitrarily small neighbourhood of $\underline{z}_0$ lying entirely within $|\underline{z}|<\underline{1}$ [10], thus violating the condition $|b/a|\leq 1$ for $|\underline{z}|<\underline{1}$.

Next, we claim that if a is not a constant, and if $a(\underline{z}_0)=0$ for some $\underline{z}_0$ with $|\underline{z}_0|=\underline{1}$, then $b(\underline{z}_0)=0$. To substantiate this claim consider the open connected set $\Omega$ of points $\underline{z}$ lying in a neighbourhood of $\underline{z}_0$ as well as inside $|\underline{z}|<\underline{1}$. Obviously, $|b/a|\leq 1$ and $a\neq 0$ for $\underline{z}\in\Omega$. It then follows by invoking the continuity

of the function b/a in $\Omega$ that if $a(\underline{z}_0)=0$ then also have $b(\underline{z}_0)=0$. Now, since the polynomials a and b are relatively prime the set of zeros common to them cannot, due to propertty A3, form a sequentially almost complete set of order (k-1). Thus, the set of zeros of a on $|\underline{z}|=\underline{1}$ does not form a sequentially almost complete set of order (k-1). It then follows by invoking theorem 2.2.3.8 that a is a scattering Schur polynomial.

<u>Corollary 2.2.3.13:</u> Let the rational function $\rho=b/a$ be such that a is widest sense Schur and $|\rho|\leq 1$ for all those $\underline{z}$ on $|\underline{z}|=1$ for which $a\neq 0$. Then, if $\rho=b_0/a_0$ in irreducible form then $a_0$ is scattering Schur.

<u>Proof:</u> Follows immediately from lemma 2.2.3.11 and theorem 2.2.3.12.

## 2.3. ELEMENTARY PROPERTIES OF (k-D) RATIONAL DISCRETE POSITIVE FUNCTIONS:

Definition 2.3.1: A rational function $\zeta$ will be called discrete positive if Re $\zeta \geq 0$ for $|\underline{z}|<\underline{1}$.

A discrete positive function $\zeta = jC$, where C is a real constant, is said to be trivial. All other discrete positive functions are said to be non-trivial.

Definition 2.3.2: A discrete positive function $\zeta$=b/a with a, b polynomials, will be said to be a discrete reactance function, if in addition, it satisfies: $(b/a) + (\tilde{b}/\tilde{a}) = 0$.

Lemma 2.3.3: let $\zeta' = \zeta'(\underline{z}')$ be the well defined rational function in (k-1) variables obtained by freezing in the k-variable rational function $\zeta = \zeta(\underline{z})$ the variable $z_1$ on $|z_1|=1$. Then if $\zeta$ is discrete positive real so is $\zeta'$.

Proof: From definition 2.3.1 via the use of arguments similar to those used in the proof of lemma 2.5.11.

Theorem 2.3.4: A discrete positive function $\zeta$ is non-trivial if and only if Re$\zeta > 0$ for $|\underline{z}|<\underline{1}$.

Proof: If $\zeta$ is a trivial positive function, then Re$\zeta = 0$ for all $\underline{z}$, and in particular for $|\underline{z}|<\underline{1}$. On the otherhand, if $\zeta$ involves one of the variables, say $z_1$, then let us freeze all other variable $\underline{z}' = (z_2, z_3, \ldots z_k)$ at $\underline{z}' = \underline{z}_0'$ in $|\underline{z}'|<\underline{1}$ in $\zeta$ and let the resulting function be called $\zeta'$. Then $\zeta'$, due to lemma 2.3.3, is a non-trivial discrete positive function of one variable $z_1$ only, for which it is well known that Re$\zeta'$>0 for $|z_1|<1$. Since $\underline{z}_0'$ is chosen arbitrarily in $|\underline{z}|<\underline{1}$, the present theorem is established.

Theorem 2.3.5: If $\zeta$=b/a, with a and b polynomials, is a discret positive function in irreducible form then both a and b are widest sense Schur polynomials. Furthermore, neither a nor b may have selfparaconjugate Schur factors of multiplicity larger than one.

Proof: Consider the nonconstant rational function $\rho = (1-\zeta)/(1+\zeta) =$

$(a-b)/(a+b)$, which is necessarily in irreducible form. We have $\mathrm{Re}\zeta \geq 0$, and consequently $|\rho| \leq 1$ for $|\underline{z}| < \underline{1}$. Therefore, due to theorem 2.2.3.12(b), $(a+b)$ is scattering Schur. Thus, $\rho$ is holomorphic in $|\underline{z}| < \underline{1}$, and consequently, the maximum modulus theorem for functions of several complex variables [9] implies that $|\rho| < 1$ in $|\underline{z}| < \underline{1}$. Now, if $b(z_0) \neq 0$ but $a(z_0) = 0$ or $b(z_0) = 0$ but $a(z_0) \neq 0$ with $|\underline{z}_0| < \underline{1}$ then we would obviously have $|\rho(\underline{z}_0)| = 1$, which is impossible. Furthermore, if for some $\underline{z}_0$ with $|\underline{z}_0| < \underline{1}$, $a(z_0) = b(z_0) = 0$, then $\rho$ would have a nonessential singularity of the second kind at $z = z_0$. However, $\rho$ would then also have a singularity of first kind in an arbitrarily small neighbourhood of $\underline{z} = \underline{z}_0$ contained in $|\underline{z}| < \underline{1}$ [10], which is ruled out by $|\rho| < 1$.

To prove the second part, let $b = e.f^n$ where $e,f$ are polynomials such that $f$ is discrete selfparaconjugate and does not divide $e$. Then due to theorem 2.2.1.3 there exists a sequentially almost complete set $\Theta'$ of $(k-1)$ tuples of order $(k-1)$ such that the one variable polynomial $b' = b'(z_1)$ in the variable $z_1$ obtained by freezing in $f$ the remaining variables at $z_i = \exp(j\theta_i)$, $i = 2$ to $k$ with $\underline{\theta}' = (\theta_2, \theta_3, \ldots \theta_k) \varepsilon \Theta'$ is also widest sense Schur, where $\Theta'$ is a sequentially almost complete set of order $(k-1)$. Let $a'$ and $f'$ be the polynomials obtained by freezing respectively in $a$ and $f$ the variables $z_i$ at $z_i = \exp(j\theta_i)$ for $i = 2$ to $k$. Then, since the pair $(a,f)$ is coprime, due to property 2.A4 in the appendix, there exists a sequentially almost complete set $\Theta''$ of $(k-1)$ tuples of order $(k-1)$ such that for any choice of $\underline{\theta}' = (\theta_2, \theta_3, \ldots \theta_k) \varepsilon \Theta''$, the pair $(a', f')$ is also coprime. Therefore, if $\underline{\theta}' \varepsilon (\Theta' \cap \Theta'')$ then the one variable rational function $\zeta'(z_1) = b'(z_1)/a'(z_1)$ is well defined, and in its irreducible form the numerator polynomial $b'$ contains the factor $f' = f'(z_1)$ of multiplicity $n$. Note that the polynomial $f'$, due to theorem 2.2.2.3, is selfparaconjugate Schur. Furthermore, due to lemma 2.3.4, $\zeta'$ is a discrete positive function in one variable, and is thus devoid of multiple zeros on the unit circle $|z_1| = 1$. Consequently $n = 1$.

<u>Assertion 2.3.6:</u> Let $\zeta = b/a$ be a discrete positive function. Then $\zeta$ is a reactance function if and only if $\deg_i b = \deg_i a$ for all $i = 1$ to $k$ and $b/a = -\hat{b}/\hat{a}$.

<u>Proof:</u> let $\zeta = d/c$ in irreducible form, where $b = ed$ and $a = ec$, $e$ being the g.c.d between $a$ and $b$. Then due to property 2.B1 in the appendix, $b/a = \tilde{b}/\tilde{a}$

implies $\tilde{d}/\tilde{c} = -\hat{d}/\hat{c} = -(\hat{d}/\hat{c}) \prod_{i=1}^{k} z^{n_{ci}-n_{di}}$, where $n_{ci} = \deg_i c$, $n_{di} = \deg_i d$ for $i = 1$ to $k$. Also, due to property 2.B2 in the appendix $\hat{c}$, and $\hat{d}$ does not contain a factor $z_i$ for any $i$. Furthermore, c and d, being numerator and denominator of irreducible discrete positive function, cannot have a zero in $|\underline{z}|<\underline{1}$ due to theorem 2.3.5, and thus cannot have a factor $z_i$ for any i. Consequently, $n_{ci} = n_{di}$ for all i, therefore, $\deg_i b = \deg_i \hat{e} + n_{di} = \deg_i \hat{e} + n_{ci} = \deg_i \hat{a}$ for all $i = 1$ to $k$. Furthermore, $\hat{b}/\hat{a} = \tilde{d}/\tilde{c} = -\hat{d}/\hat{c} = -(\hat{d}.\hat{e})/(\hat{c}.\hat{e}) = -\hat{b}/\hat{a}$, where the last equality follows from property 2.B1 in the appendix. Conversely, if $\deg_i b = \deg_i a$ for $i = 1$ to $k$, then obviously $\hat{b}/\hat{a} = \tilde{b}/\tilde{a}$. Thus, if $\zeta = b/a$ is a discrete positive function with $\hat{b}/\hat{a} = -\tilde{b}/\tilde{a}$, we would also have $b/a + \tilde{b}/\tilde{a} = 0$ implying that $\zeta$ is a reactance function.

Notice, however that a discrete positive function can have a non-zero difference in partial degree between its numerator and denominator. Consider, $\zeta = 1+[(1-z_1z_2)/(1+z_1z_2)] = [1+z_1z_2]^{-1}$, which is a discrete positive function.

<u>Theorem 2.3.7:</u> If a rational function $\zeta = b/a$ satisfies: (i) a is scattering Schur (ii) Re $\zeta \geq 0$ for $|\underline{z}|=\underline{1}$, whenever $a \neq 0$ then $\mathrm{Re}\zeta \geq 0$ for $|\underline{z}| < \underline{1}$ i.e., $\zeta$ is a discrete positive function.

<u>Proof:</u> Let us assume for the purpose of induction that the theorem be true for (k-1) variables. Also, let the polynomial $a'=a'(\underline{z}')=a(z_{10},\underline{z}')$ and the rational function $\zeta'=\zeta'(\underline{z}')=\zeta(z_{10},\underline{z}')$ be defined by freezing respectively in a and in $\zeta$ the variable $z_1$ at $z_{10}$ on $|z_1|=1$. Since a is scattering Schur, due to theorem 2.2.3.4, a' is scattering Schur for an almost complete set of values $z_{10}$ on $|z_1|=1$. For those $\underline{z}'$ on $|\underline{z}'|=\underline{1}$ for which $a' \neq 0$, we have $a(z_{10},\underline{z}') \neq 0$, and consequently, due to condition (b) of the present theorem, $\mathrm{Re}\zeta'=\mathrm{Re}\zeta(z_{10},\underline{z}') \geq 0$. Then due to te induction hypothesis it follows that $\mathrm{Re}\zeta' \geq 0$ for $|\underline{z}'|<\underline{1}$ for an almost complete choice of $z_{10}$ on $|z_1|=1$.

Consider next $a_1=a_1(z_1)=a(z_1,\underline{z}_0')$ and $\zeta_1=\zeta_1(z_1)=\zeta(z_1,\underline{z}_0')$, where $\underline{z}_0'$ is frozen in $|\underline{z}_0'|<\underline{1}$. Then from the conclusions of the last paragraph it follows that the one-variable rational function $\zeta_1$ satisfies $\mathrm{Re}\zeta_1 \geq 0$ for an almost complete set of values on $|z_1|=1$. Furthermore, $a_1$, due to theorem 2.2.3.10 and corollary 2.2.3.3, is a strict sense Schur polynomial in $z_1$ only. Therefore, from a classical one-variable result it follows that $\mathrm{Re}\zeta_1 \geq 0$ for $|z_1|<1$. Since $\underline{z}_0'$ is

arbitrarily chosen in $|\underline{z}'|<\underline{1}$, the last condition yields that $\mathrm{Re}\zeta\geq 0$ for $|\underline{z}|<\underline{1}$, thus completing the proof of the present theorem.

Remark 2.3.8: The above theorem, in the one variable case, can be obtained from an application of lemma 2.2.3.11 with the nonrational function $\exp(-\zeta)$. However, since a multidimensional version of lemma 2.2.3.11 valid for nonrational functions is not known to us, a separate proof of theorem 2.3.8 is needed.

Theorem 2.3.9: If $\zeta=b/(ac)$ is a discrete positive function in k-variables $\underline{z}$, where a is a scattering Schur polynomial and c is a reactance Schur polynomial in one variable only then $\zeta$ can be decomposed as $\zeta=(c'/c)+(a'/a)$, where $c'/c$ is a one variable discrete reactance function and $a'/a$ is a discrete positive function in k-variables $\underline{z}$.

Proof: Let $c=\sum_{i=1}^{n} c_i$, where $c_i=(z_i-\alpha_i)$ with $|\alpha_i|=1$ and $\alpha_i$'s are necessarily distinct. Consider the rational function $K_\nu(\underline{z}') = [(z_1-\alpha_\nu)b/ac]_{z_1=\alpha_\nu}$, which is obviously holomorphic in $|\underline{z}'|<\underline{1}$. Furthermore, for all $\underline{z}_0'$ with $|\underline{z}_0'|<\underline{1}$, $K_\nu(\underline{z}_0')$ is the residue of the one-variable discrete positive function $\zeta_1=\zeta_1(z_1)=\zeta(z_1,\underline{z}_0')$ at $z_1=\alpha_\nu$, and is thus real and positive. Since a function of several complex variables, which is real and holomorphic in a domain must be equal to a constant we have that $K_\nu(\underline{z}')=K_\nu=$constant. Also, since the polynomial $h=h(\underline{z})=b-a\sum_{i=1}^{n} K_i c/c_i$ satisfies $h(\alpha_\nu,\underline{z}')=0$, $(z_1-\alpha_\nu)$ must be a factor of h for each $\nu$ i.e., $h=ca'$ for some polynomial $a'=a'(\underline{z})$. It then follows by straightforward algebraic manipulation that $\zeta=(c'/c)+(a'/a)$, where $c'=\sum_{i=1}^{n} K_i c/c_i$. Clearly, $c'/c$ is a one-variable discrete reactance function, and therefore, $\mathrm{Re}(c'/c)=0$ for $|z_1|=1$, whenever $c\neq 0$. Since $\zeta$ is a discrete positive function, $\mathrm{Re}\zeta\geq 0$ for $|\underline{z}|=1$, whenever $\zeta$ is regular. Therefore, $\mathrm{Re}(a'/a)\geq 0$ for $|\underline{z}|=1$, whenever $a\neq 0$. It then follows via the use of theorem 2.3.8 that $a'/a$ is a discrete positive function.

Theorem 2.3.10: Let $\zeta=b/a$, where a and b are relatively prime polynomials. Then $\zeta$ is a discrete positive function if and only if (a+b) is scattering Schur, and $\mathrm{Re}\zeta\geq 0$ for $|\underline{z}|=1$, wherever $\zeta$ is regular.

**Proof:** Consider the rational function $\rho=(1-\zeta)/(1+\zeta)=(a-b)/(a+b)$ in irreducible form and notice that $\mathrm{Re}\,\zeta\geq 0$ if and only if $|\rho|\leq 1$.

Necessity: Since $\mathrm{Re}\,\zeta\geq 0$ for $|\underline{z}|<\underline{1}$ we have $|\rho|\leq 1$ for $|\underline{z}|\leq 1$. Therefore, due to theorem 2.2.3.12(b), $(a+b)$ is scattering Schur. Finally, the fact that $\mathrm{Re}\,\zeta\geq 0$ for all regular points of $\zeta$ on $|\underline{z}|=\underline{1}$ follows from lemma 2.3.3.

Sufficiency: If $a+b\neq 0$ but $a=0$ for some $\underline{z}$ then $|\rho|=1$. On the other hand, if $a+b\neq 0$ but $a\neq 0$ for some $\underline{z}=\underline{z}_0$ with $|\underline{z}_0|=\underline{1}$ then $\rho$ is regular at $\underline{z}_0$, and consequently, $\mathrm{Re}\,\zeta(\underline{z}_0)\geq 0$, thus implying $|\rho(\underline{z}_0)|\leq 1$. Therefore, $|\rho|\leq 1$ for $|\underline{z}|=1$, whenever $a+b\neq 0$ i.e., $\rho$ is holomophic. The result then follows from lemma 2.2.3.11 via the use of the fact that $(a+b)$ is scattering Schur.

We have the following important result.

**Theorem 2.3.11:** If a is a widest sense Schur polynomial involving one of the k variables, say $z_i$, then $a_{z_i}/a$ is a discrete positive function.

**Proof:** Assume $i = 1$. Since a is widest sense Schur the polynomial $\bar{a}$ in one variable $z_1$ obtained by freezing in a the variables $z_i, i=2$ to k inside $|z_i|< 1$, is widest sense Schur. The result then follows from property 2.C3 in the appendix by noting that $\mathrm{Re}(\bar{a}_{z_1}/a) = \mathrm{Re}(a_{z_1}/a)>0$ for $|\underline{z}|<\underline{1}$.

**Theorem 2.3.12:** If a is a selfparaconjugate Schur polynomial involving one of the k variables, say $z_i$, then $a_{z_i}/a$ is a discrete reactance function (not necessarily in irreducible form).

**Proof:** Assume $i = 1$, and $\hat{a} = \gamma a$. The fact that $a_{z_1}/a$ is a discrete positive function has been proved in theorem 2.3.11. If a is expressed as a polynomial in $z_1$ with coefficients as polynomials in $\underline{z}'$ i.e., in the form (2.1.1) with $i = 1$, then $a_{z_1} = \sum_{\nu=0}^{n_1} b_\nu z_1^\nu$, where $b_\nu = (n_1-2\nu)a_\nu$ for $\nu=0,1,\ldots n_1$. Furthermore, from lemma 2.2.2.9 we have that $\deg_i a_{z_1} = \deg_i a = n_i$ for $i = 1$ to k. Therefore, $\hat{a}_{z_1}=(\sum_{\nu=0}^{n_1}\tilde{b}_\nu z_1^{-\nu})\prod_{i=1}^{k} z_i^{n_i} = (\sum_{\nu=0}^{n_1} -(n_1-2\nu)\tilde{a}_{n_1-\nu} z^\nu)(\prod_{i=2}^{k} z_i^{n_i})$ (via the substitution $\tilde{b}_\nu = (n_1-2\nu)\tilde{a}_\nu$) $= \sum_{\nu=0}^{n_1} -(n_1-2\nu)\gamma a_\nu z^\nu$ (via the use of equation (2.A1)) $=-\gamma\sum_{\nu=0}^{n_1} b_\nu z^\nu=$

$-\gamma a_{z_1}$. Therefore, $(a_{z_1}/a) + (\hat{a}_{z_1})/\hat{a} = 0$. The fact that $a_{z_1}/a$ is a discrete reactance function then following by invoking assertion 2.3.6.

Corollary 2.3.13: If a is an irreducible widest sense (discrete selfparaconjugate) Schur polynomial involving the variable $z_i$ then $a_{z_i}/a$ is a discrete positive (reactance) function in irreducible form.

Proof: Immediately follows from theorems 2.3.11, 3.12 and property 2.C2 in the appendix.

Theorem 2.3.14: If a is a widest sense Schur polynomial involving the variable $z_i$ then the polynomial $a_{z_i}$ is also widest sense Schur.

Proof: let $a_{z_i}/a = c/b$, $a = bd$ and $a_{z_i} = cd$, d being the g.c.d between $a_{z_i}$ and a. Then due to theorem 2.3.11, c/b is discrete positive and consequently, due to theorem 2.3.5, b and c are widest sense Schur polynomials. Furthermore, the polynomial d, being a factor of a, which is widest sense Schur, is also widest sense Schur. Thus $a_{z_i} = c.d$ is widest sense Schur.

Theorem 2.3.15: If $\zeta = b/a$ is a nontrivial discrete positive function in irreducible form then $f = xa + jyb$ is widest sense Schur for all real x and y not simultaneously zero. Conversely, if the polynomial $f = xa+jyb$ is widest sense Schur for all real x and y not simultaneously zero then either b/a or (-b/a) is a discrete positive function.

Proof: If b/a is a nontrivial discrete positive function in irreducible form then due to theorem 2.3.5, $a\neq0$, $b\neq0$ for $|\underline{z}|<\underline{1}$. Consequently, if $f(\underline{z}_0) = 0$ for some $\underline{z}_0$ in $|\underline{z}|<\underline{1}$ then $b/a = j(x/y)$, implying $Re(b/a) = 0$ for $\underline{z} = \underline{z}_0$ in $|\underline{z}|<\underline{1}$. The latter conclusion, in view of theorem 2.3.4 contradicts with the fact that (b/a) is a nontrivial discrete positive function. Conversely, if f is widest sense Schur for all real x and y except $x = y = 0$, then a and b are also widest sense Schur, because if $a(\underline{z}_0) = 0$ (or $b(\underline{z}_0)=0$) with $|\underline{z}_0|<\underline{1}$, then $f(\underline{z}_0) = 0$ for $y = 0$(or $x = 0$) and for arbitrary x(or y). Thus, Re(b/a) is a continuous function of $\underline{z}$ in $|\underline{z}|<\underline{1}$. Furthermore, $Re(b/a) \neq 0$ for any $\underline{z}$ in $|\underline{z}|<\underline{1}$, because otherwise $b/a = -jy/x$,

i.e., $xa+jby = 0$ for some real $x$ and $y$ not simultaneously zero. Thus, by continuity of $b/a$ in $|\underline{z}|<\underline{1}$, if $\mathrm{Re}(b/a)>0$ (or $<0$) for some $\underline{z}$ in $|\underline{z}|<1$, then $\mathrm{Re}(b/a)>0$ (or $<0$) for all $|\underline{z}|<\underline{1}$.

Corollary 2.3.16: The rational function $\zeta = b/a$, where $b$ and $a$ are relatively prime polynomials, is a nontrivial discrete positive function if and only if $(x^2a^2+y^2b^2)$ is a widest sense Schur polynomial for all real $x$ and $y$ except $x = y = 0$.

Proof: Since $(x^2a^2+y^2b^2)=(xa+jyb)(xa-jyb)$, the above corollary clearly follows from theorem 2.3.15 and the obvious fact that a polynomial is widest sense Schur if and only if its factors are widest sense Schur.

We note that the above result is a discrete k-D counterpart of a result originaly proved by Brockett [11] in the 1-D context. The k-D continuous counterpart of corollary 2.3.16 is obtained in [5], however, via a proof technique entirely different from the one presented here.

## 2.4. THE REACTANCE SCHUR AND IMMITTANCE SCHUR POLYNOMIALS:

<u>Definition 2.4.1:</u> A selfparaconjugate Schur polynomial $a$ is said to be reactance Schur if the irreducible factors of $a$ do not occur with multiplicity larger than one.

The following results immediately follows from the above definition.

<u>Theorem 2.4.2:</u> (a) Products of reactance Schur polynomials that are pairwise relatively prime are reactance Schur polynomials. (b) Any factor of a reactance Schur polynomial $a$ is also a reactance Schur polynomial.

<u>Proof</u>: (a) Follows from the fact that the product of selfparaconjugate Schur polynomials is also selfparaconjugate Schur. (b) We note that definition 2.4.1 and theorem 2.2.2.8 imply that the irreducible factors of $a$ are selfparaconjugate Schur, and are thus reactance Schur. The result then follows by appealing to part (a) of the present theorem.

<u>Theorem 2.4.3:</u> (a) The numerator and denominator of a discrete reactance function, written in irreducible form, are always reactance Schur polynomials. (b) Conversely, each discrete reactance Schur polynomial $b$ is the denominator (and consequently the numerator) of a discrete reactance function in irreducible form.

<u>Proof</u>: (a) If $\zeta = b/a$ is discrete reactance function, and in particular a discrete positive function in irreducible form, then due to theorem 2.3.5, $a$ is a widest sense Schur polynomial devoid of multiple discrete selfparaconjugate factors. Furthermore, since $a$ is a widest sense Schur it cannot contain $z_i$ as a factor for any $i$. Thus, $\deg_i \hat{a} = \deg_i a$ for $i = 1$ to $k$. Since from assertion 2.3.6 we have $b/a = -\hat{b}/\hat{a}$, in irreducible form, we must have $\hat{a} = \gamma a$, where $\gamma$ is a constant i.e., $a$ is selfparaconjugate Schur. Since irreducible factors of $a$, due to theorem 2.2.2.8, are also selfparaconjugate Schur, they must occur in $a$ with multiplicity equal to one. Therefore, $a$ is reactance Schur. Similar arguments apply to $b$.

(b) Let $b_i$, $i = 1,2,\ldots n$ be the nontrivial irreducible factors of $b$, which are also reactance Schur due to theorem 2.4.2. Let $b_i$ involve the variable $z_{\lambda(i)}$

and consider the rational function $\phi = \sum_{i=1}^{n} \phi_i$, where $\phi_i = (b_i)_{\lambda(i)}/b_i$. Since from theorem 2.3.12, each $\phi_i$ is a discrete reactance function, their sum $\phi$ is also a discrete reactance function. Let $\phi = c/b$, then $c = \sum_{i=1}^{n} ((b_i)_{\lambda(i)} \prod_{\substack{\nu=1 \\ \nu \neq i}}^{n} b_\nu)$. It then follows by appealing to property 2.C2 in the appendix that c and b are relatively prime polynomials.

Theorem 2.4.4: If the polynomial a is nontrivial and is scattering Schur, then the discrete paraeven and the discrete paraodd parts of a respectively denoted by $a_e=(a+\hat{a})/2$ and $a_o=(a-\hat{a})/2$ are relatively prime reactance Schur polynomials.

Proof: Since a is scattering Schur the rational function $\rho=\hat{a}/a$, due to definition 2.2.3.1, is in irreducible form and satisfies $|\rho|\leq 1$ for $|\underline{z}|<\underline{1}$. Therefore, if $\zeta = (1-\rho)/(1+\rho) = (a-\hat{a})/(a+\hat{a})$, then $\zeta$ is a discrete positive function (i.e. $\mathrm{Re}\zeta \geq 0$ for $|\underline{z}|<\underline{1}$) in irreducible form. Consequently, due to theorem 2.3.5, both $a_e = a+\hat{a}$ and $a_0 = a - \hat{a}$ are widest sense Schur. The facts that $a_e$ and $a_0$ are respectively paraeven and paraodd follows from property 2.B10. Therefore, $a_e$ and $a_0$ are selfparaconjugate Schur. Furthermore, by appealing to the second part of theorem 2.3.5 it follows that irreducible factors of $a_e$ and $a_0$, which by virtue of theorem 2.2.2.8, must be selfparaconjugate Schur, must not occur with multiplicity higher than one. Therefore, $a_0$ and $a_e$ are reactance Schur polynomials.

It is now possible to derive the converse of theorem 2.4.4.

Theorem 2.4.5: If b is a discrete paraeven (or discrete paraodd) reactance Schur polynomial then there exists a discrete paraodd (or discrete paraeven) reactance Schur polynomial c, relatively prime with b, such that (b+c) is either a nonzero constant or a nontrivial scattering Schur polynomial.

Proof: Consider the irreducible discrete reactance function $\zeta = c/b$, the existence of which has been demonstrated in theorem 2.4.3(b). Since due to assertion 2.3.6 we have $(c/b) + (\hat{c}/\hat{b}) = 0$, $\hat{b} = \pm b$ implies $\hat{c} = \mp c$. Thus, if b is paraeven (or paraodd) then c is paraodd (or paraeven). Furthermore, the rational function $\rho = (1-\zeta)/(1+\zeta) = (b-c)/(b+c)$ is irreducible form, and satisfies $|\rho|<\leq 1$ for $|\underline{z}|<\underline{1}$. Consequently, due to theorem 2.2.3.12(b), (b+c) is either a nonzero constant or scattering Schur polynomial.

In the above theorem it is indeed possible to have b+c = constant. Consider $b = 1 + z_1 z_2$, $c = 1 - z_1 z_2$.

The following is an yet alternate characterization of reactance Schur polynomials.

<u>Theorem 2.4.7:</u> Let a be a selfparaconjugate Schur polynomial. Then a is reactance Schur if and only if the set of (k+1) polynomials $a, a_{z_1}, a_{z_2}, \ldots a_{z_k}$ do not have a proper common factor.

<u>Proof</u>: In view of property 2.C2 in the appendix, it follows that the irreducidble factors of a are simple if and only if the set of (k+1) polynomials $a, a_{z_1}, \ldots a_{z_k}$ do not have a proper common factor. The result then follows by appealing to the definition 2.4.1 of reactance Schur polynomials.

<u>Definition 2.4.8:</u> A polynomial a is said to be immittance Schur if it is the product of a scattering Schur polynomial and a reactance Schur polynomial.

The following result clearly follows.

<u>Theorem 2.4.9:</u> (a) Factors of immittance Schur polynomials are immittance Schur. (b) Products of immittance schur polynomials that do not have a discrete selfparaconjugate factor in common are also immittance Schur. (c) The least common multiple of a set of immittance Schur polynomials is also immittance Schur.

The following result is an obvious consequence of theorems 2.3.5.

<u>Theorem 2.4.10:</u> The numerator and denomiantor polynomials a and b of a discrete positive function $\zeta = a/b$ in irreducible form are necessarily immittance Schur.

<u>Theorem 2.4.11:</u> If a is a polynomial involving $z_i$ and is immittance Schur then the polynomial $a_{z_i}$ is also immittance Schur.

<u>Proof</u>: Let $a_{z_i}/a = b/c$, with $a_{z_i} = b.d$ and $a = c.d$, where d is the gcd between $a_{z_i}$. Due to theorem 2.4.9, d is immittance Schur. Since the polynomials b and c are relatively prime, the rational function b/c, due to theorem 2.3.5 is a

discrete positive function in irreducible form. Consequently, by virtue of theorem 2.4.10, b is an immittance Schur polynomial. Let d contain a selfparaconjugate Schur factor e. Since e is a factor of a, which is immittance Schur, e occurs in a with multiplicity one. Now, if e involves $z_i$ then $a_{z_i}$, by property 2.C2, does not contain the factor e, which contradicts $a_{z_i} = b.d$. Therefore, d cannot contain e, and in particular, b and d do not have discrete selfparaconjugate factors involving $z_i$ in common. On the other hand, if e does not involve $z_i$ then the multiplicity of e in a and $a_{z_i}$ must be the same. Also, since a=c.d is immittance Schur, c cannot contain a factor e. Thus, b cannot contain the factor e, beacuse otherwise the multiplicity of e in $a_{z_i}$ would be larger than in a. Therefore, b and d cannot have a discrete selfparaconjugate factor not involving $z_i$ in common. Therefore, b and d cannot have a common selfparaconjugate Schur factor, thus confirming the immittance Schur property of $a_{z_i} = b.d$ in view of theorem 2.4.9.

<u>Theorem 2.4.12:</u> The denominator of a rational discrete positive function in irreducible form is an immittance Schur polynomial. Conversely, every immittance Schur polynomial is the denominator of a discrete positive function in irreducible form.

<u>Proof</u>: The first half of the present theorem has already been proved in theorem 2.4.10. To prove the second half let a = e.f, where e and f are respectively the scattering Schur and reactance Schur factors of a (cf. theroem 2.2.6). Notice that e and f are relatively prime due to theorem 2.2.2.8 and 2.2.3.2. Since e is scattering Schur, theorem 2.2.3.12(a) yields $|\hat{e}/e|\le 1$ for $|\underline{z}|<\underline{1}$ implying $Re[1+(\hat{e}/e)]\ge 0$ for $|\underline{z}|<\underline{1}$ i.e., $[1+(\hat{e}/e)]$ is a discrete positive function. Furthermore, since f is reactance Schur, by virtue of theorem 2.4.3, there exists a polynomial g, such that g/f is a discrete reactance function (and thus a positive function) in irreducible form. Therefore, $1+(\hat{e}/e)+(g/f) = p/a$, where $p = e(f+g)+\hat{e}f = (e+\hat{e})f+eg$ is a discrete positive function. It remains to show that p and a=ef are relatively prime polynomials. Clearly, if e and p had a factor in common, then e and $(\hat{e}f)$ would have a factor in common. However, e, being scattering Schur, is relatively prime with $\hat{e}$, and cannot have a factor in common with f as stated earlier. On the other hand, if f had a factor in common with p then it would have a factor in common with eg. Since g/f is an irreducible rational function, this later situation implies that e

and $f$ would have a factor in common, which is again impossible. Thus, $p/a$ is an irreducible rational function.

## 2.5. EXTENSIONS TO PASSIVE MULTIPORT SCATTERING AND IMMITTANCE FUNCTIONS:

The following results generalize the above discussion on the scattering or immittance description of k-D discrete passive multidimensional systems to multiport systems.

Definition 2.5.1: An (NxN) rational matrix $H=H(\underline{z})$ is said to be discrete bounded if H is holomorphic in $|\underline{z}|<\underline{1}$ and $(I_N-HH^{*t})$ is nonnegative definite at all regular points of H in the domain $|\underline{z}|\leq\underline{1}$.

We shall need the following extensions of lemma 2.2.3.11 for further developments to follows. Let us consider the function $\rho=\rho(\underline{z})=\sum_{i=1}^{N}|\rho_i(\underline{z})|$, where each $\rho_i=b_i(\underline{z})/a_i(\underline{z})$ with $a_i=a_i(\underline{z})$, $b_i=b_i(\underline{z})$ polynomials in k-variables $\underline{z}$. We note that since $\rho$ is not an analytic function of $\underline{z}$ neither the maximum modulus principle for functions of several complex variables [9] nor lemma 2.2.3.11 of the present paper applies to $\rho$. However, the following result is true.

Lemma 2.5.2: Let $\rho=\rho(\underline{z})$ be as defined above. If $a_i=a_i(\underline{z})$ for each i=1,2...N are widest sense Schur polynomials and $\rho\leq 1$ for $|\underline{z}|=1$, whenever $a_i\neq 0$ for each i=1,2...N, then $|\rho|\leq 1$ for $|\underline{z}|<\underline{1}$. Furthermore, if $\rho=1$ for some $\underline{z}$ in $|\underline{z}|=\underline{1}$ then $\rho<1$ for $|\underline{z}|<\underline{1}$ unless each $\rho_i$ is constant.

Proof: Let $\rho(\underline{z}_0)\geq 1$ with $|\underline{z}_0|<\underline{1}$. Consider the rational functions $r_i(\underline{z})=C_i\rho_i(\underline{z})$, i=1,2...N where each $C_i$ is a constant such that $|C_i|=1$, and $r_i(\underline{z}_0)$ is a positive real number i.e., $|\rho_i(\underline{z}_0)|=|r_i(\underline{z}_0)|=r_i(\underline{z}_0)$. Also, consider the rational function $R(\underline{z})=\sum_{i=1}^{N} r_i(\underline{z})$. Obviously, then $|R(\underline{z})|\leq\sum_{i=1}^{N}|r_i(\underline{z})|=\sum_{i=1}^{N}|\rho_i(\underline{z})|=\rho(\underline{z})$ for all $\underline{z}$. In particular, $|R(\underline{z})|\leq\rho(\underline{z})\leq 1$ for $|\underline{z}|=\underline{1}$, whenever $a_i\neq 0$ for each i=1,2...N. Thus, by invoking lemma 2.2.3.11 with $R(\underline{z})$ it follows that either $|R(\underline{z})|<1$ for $|\underline{z}|<\underline{1}$ or $R(\underline{z})$ is a constant. The first of the last two situations is, however, in contradiction with the fact that $R(\underline{z}_0)=\sum_{i=1}^{N} r_i(\underline{z}_0)=\sum_{i=1}^{N}|\rho_i(\underline{z}_0)|=\rho(\underline{z}_0)\geq 1$. On the other hand, if $R(\underline{z})$ is a contant then it follows that $1\leq|\rho(\underline{z}_0)|=|R(\underline{z}_0)|=|R(\underline{z})|\leq\rho(\underline{z})$ for all $\underline{z}$. However, since $|\rho(\underline{z})|\geq 1$ for $|\underline{z}|=1$ whenever $a_i\neq 0$ for each i, the last stated chain of weak inequalities, in fact hold with inequalities replaced by equalities. Thus, we have that $\rho(\underline{z})=1$ for all $\underline{z}$. Furthermore, a comparison of the equality

$\sum_{i=1}^{N} |r_i(\underline{z})| = \sum_{i=1}^{N} \rho_i(\underline{z})| = |\rho(\underline{z})| = |R(\underline{z})|$ with $\sum_{i=1}^{N} r_i(\underline{z}) = R(\underline{z})$ clearly shows that $r_i(\underline{z})$ for each $i=1,2\ldots N$ is a real positive number for all $\underline{z}$. It then follows by invoking a standard result from the theory of function of several complex variables that $r_i(\underline{z})$, for each i, is a constant i.e., independent of $\underline{z}$.

The following result can be viewed as a multiport extension of the maximum modulus theorem for function of several compex variables useful in the present context of passive or lossless networks.

Theorem 2.5.3: Let H be an (NxN) rational matrix holomorphic in $|\underline{z}|<\underline{1}$ and $(I_N - HH^{*t})$ is nonnegative definite for $|\underline{z}|=\underline{1}$, wherever H is well defined. Then H is a discrete bounded matrix.

Proof: Let $x=(x_1,x_2,\ldots x_k)$ be any (1xN) constant vector and $y=(y_1,y_2,\ldots y_k)$ be a (1xN) vector defined via $y=Hx$. Then it follows from the nonnegative definiteness of $(I_N - HH^{*t})$ that $\sum_{i=1}^{N} |y_i|^2 \leq \sum_{i=1}^{N} |x_i|^2$, whenever H and thus, each $y_i = y_i(\underline{z})$ is well defined on $|\underline{z}|=1$. Clearly, the property of holomorphy in $|\underline{z}|<\underline{1}$ is inherited by each $y_i$ from H. Then by invoking lemma 2.5.2, it follows that $\sum_{i=1}^{N} |y_i|^2 \leq \sum_{i=1}^{N} |x_i|^2$ for $|\underline{z}|<\underline{1}$. Since x is arbitrary the last conclusion yields that $(I_N - HH^{*t})$ is nonnegative definite for $|\underline{z}|<\underline{1}$, which when combined with the fact that $(I_N - HH^{*t})$ is nonnegative definite whenever H is holomorphic in $|\underline{z}|=\underline{1}$, yields, in view of definition 2.5.1, that H is a discrete bounded matrix.

Definition 2.5.4: A rational matrix of size (NxN) is said to be discrete k-D lossless bounded if

(a) the entries $H_{ij} = H_{ij}(\underline{z})$ of $H = H(\underline{z})$ are holomorphic and unimodularly bounded i.e., $|H_{ij}| \leq 1$ for $|\underline{z}| < \underline{1}$.

(b) $H\tilde{H} = I_N$.

We note that, as expected, the class of multidimensional discrete lossless bounded matrices form a subclass of the class of multidimensional discrete bounded matrices.

Property 2.5.5: The denominators of each entry of discrete k-D lossless bounded matrix H, when written in irreducible form, are scattering Schur polynomials.

Proof: Follows from theorem 2.2.3.12(b).

Property 2.5.6: Determinant of H can be expressed in irreducible form as det H = $d(a/\hat{a})$, where a is a scattering Schur polynomial, and d is a unimodular constant i.e., $|d| = 1$.

Proof: Let det H = $a/h$, where a and h are relatively prime polynomials. Since due to theorem 2.2.3.9, products as well as factors of scattering Schur polynomials are also scattering Schur, from property 2.5.5 it follows that h is a scattering Schur polynomial. Now, from definition 2.5.4(b) of lossless bounded matrices it obviously follows that det H det $\tilde{H}$ = 1 i.e., $(a/h)(\tilde{a}/\tilde{h})=1$, thus implying (2.5.1a) below. Since in (2.5.1a), the polynomial h is relatively prime with a, and h, being scattering Schur, cannot have a factor $z_i$ for any i, the polynomial $\hat{a}$ must contain h as a factor i.e., (2.5.1b) follows, where d is a polynomial in $\underline{z}$.

$$a\,\hat{a}\,\underline{z}^{n_h} = h\hat{h}\,\underline{z}^{n_a}, \qquad \hat{a} = h.d \qquad (2.5.1a,b)$$

Substitutuing (2.5.1b) in (2.5.1a) and considering the discrete paraconjugate of the resulting equation it follows from Property 2.A5 that $\hat{a}\hat{d}=\hat{h}$. The latter equality, in view of scattering Schur property of h and property 2.A6, imply (2.5.2).

$$\hat{a}\,.\,\hat{d} = h \qquad (2.5.2)$$

Substituting for h from (2.5.2) into (2.5.1b), we obtain $d\hat{d}=1$, which in turn imply that d is a constant. Consequently, $\hat{d}=d^*$, and $dd^*=|d|=1$. It then follows from (2.5.2) that det h= $a/h = a/(\hat{a}\hat{d}) = d(a/\hat{a})$.

Remark 2.5.7: Although $\hat{a}$ is scattering Schur a is not necessarily so. Consider H = $\underline{z}$.

Property 2.5.8: A discrete lossless k-D rational matrix H can be written as H = $P/\hat{a}$, where P = $P(\underline{z})$ is a polynomial matrix in $\underline{z}$, and a is as in property

2.5.6 above.

Proof: From condition (b) of the definition 2.5.4 and property 2.5.6 it follows that $\tilde{H} = H^{-1} = d^{*}(\text{Adj } H)(\hat{a}/a)$, which in turn imply that $H = \tilde{\tilde{H}} = d(\tilde{\hat{a}}/\tilde{a})(\text{Adj}\tilde{H})$. Furthermore, it is straightforward to verify that $\tilde{\hat{a}}/\tilde{a} = a/\hat{a}$ for any polynomial a. The last two equlities combined togather yield (2.5.3) in the following.

$$\hat{a}.H = d.a.(\text{Adj } \tilde{H}) \qquad (2.5.3)$$

Due to property 2.5.5 and theorem 2.2.3.9(b) the entries of Adj H, when written in irreducible form, have scattering Schur denominators, and therefore, are holomorphic in $|\underline{z}|<\underline{1}$. Hence in (2.5.3) the right hand side is holomorphic in $|\underline{z}|>\underline{1}$. The left hand side is either a polynomial matrix $P=P(\underline{z})$ or the entries, due to property 2.5.5 and theorem 2.2.3.9(a), have scattering Schur denominators when written in irreducible form. In the latter case, however, $\hat{a}H$, by virtue of theorem 2.2.3.6, is not holomorphic in $|\underline{z}|>\underline{1}$. Thus, $\hat{a}H=P$, where $P=P(\underline{z})$ is a polynomial matrix.

Multidimensional discrete lossless two-ports (i.e., N=2):

For N=2 by considering the fact that $\tilde{H} = H^{-1}$, (2.5.4) and (2.5.5) below follows from property 2.5.6.

$$\tilde{H}_{11} = d^{*}(\hat{a}/a)H_{22} \; ; \; \tilde{H}_{21} = - d^{*}(\hat{a}/a)H_{12} \qquad (2.5.4a,b)$$

$$\tilde{H}_{12} = - d^{*}(\hat{a}/a)H_{21} \; ; \; \tilde{H}_{22} = d^{*}(\hat{a}/a)H_{11} \qquad (2.5.5a,b)$$

We note that (2.5.4a) and (2.5.4b) are respectively identical with (2.5.5b) and (2.5.5a) via the operation '~'. Also, $\tilde{H}_{12}$ in (2.5.5a) is holomorphic in $|\underline{z}|>\underline{1}$, whereas $H_{21}$, due to property 2.5.5, must have scattering Schur denominator, and therefore, due to theorem 2.2.3.6 has singularities in $|\underline{z}|>\underline{1}$. Hence the denominator of $H_{21}$ divides $\hat{a}$, i.e., $H_{21} = c/\hat{a}$ (not necessarily in irreducible form). Therefore, from (2.5.5a) it follows that $\tilde{H}_{12} = - d^{*}(c/a)$ i.e., $H_{12} = - d(\tilde{c}/\tilde{a}) = - (\tilde{c}/\hat{a})\underline{z}^{-\hat{n}_a}$. It follows from (2.5.5b) via a similar argument that $H_{11} = b/\hat{a}$ (not necessarily in irreducible form), and $H_{22} = d(b/\hat{a})\underline{z}^{-\hat{n}_a}$. Thus, H can be written as in (2.5.6) below.

$$H = \frac{1}{\hat{a}} \begin{bmatrix} b & -d\tilde{c}\,\underline{z}^{n_a} \\ c & +d\tilde{b}\,\underline{z}^{n_a} \end{bmatrix} \tag{2.5.6}$$

Also, by considering the determinant of (2.5.6), it follows via the use of property 2.5.6 that

$$b\tilde{b} + c\tilde{c} = a\tilde{a} \tag{2.5.7}$$

Furthermore, since $H_{12} = -\,d\tilde{c}\underline{z}^{n_a}/\hat{a}$ and $H_{22} = d\tilde{b}\underline{z}^{n_a}/\hat{a}$ are holomorphic in $|\underline{z}|<\underline{1}$, it is necessary that $\deg_i c \le n_{ai}$ and $\deg_i b \le n_{ai}$ for i= 1 to k.

The following result can be considered to be the discrete multidimensional counterpart of the Belevitch canonical form for the representation of lossless two-port networks well known in classical network theory. The multidimensional continuous version of this result has recently been discussed in [2].

<u>Theorem 2.5.9:</u> A (2x2) rational matrix $H = H(\underline{z})$ is a lossless bounded matrix if and only if there exists polynomials a,b,c and a unimodular complex constant d such that:

(i) H can be written as (2.5.6)
(ii) $\hat{a}$ is a scattering Schur polynomial
(iii) a,b,c are related as in (2.5.7)
(iv) $\deg_i a \ge \deg_i b$, and $\deg_i a \ge \deg_i c$ for all i=1 to k.

<u>Proof:</u> Only the sufficiency part of the theorem needs to be proved, the neceesity being already established. We need to show that under the conditions stated in the present theorem, condition (a) and (b) of definition 2.5.4 are satisfied. Straightforward computation along with (2.5.7) and the identitity $\hat{a}\tilde{\hat{a}} = \tilde{a}a$ yields $H\tilde{H} = I_2$, which verifies condition (a) of definition 2.5.4. Furthermore, a second use of the identity $\hat{a}\tilde{\hat{a}} = \tilde{a}a$ along with (2.5.7) yields (2.5.8) below.

$$(b/\hat{a})(\tilde{b}/\tilde{\hat{a}}) + (c/\hat{a})(\tilde{c}/\tilde{\hat{a}}) = 1, \quad (b/\tilde{\hat{a}})(\tilde{b}/\hat{a}) + (c/\tilde{\hat{a}})(\tilde{c}/\hat{a}) = 1 \tag{2.5.8a,b}$$

It clearly follows from (2.5.8b) that $|\tilde{b}/\hat{a}|^2 + |\tilde{c}/\hat{a}|^2 = 1$, and thus $|\tilde{b}/\hat{a}| \leq 1$, $|\tilde{c}/\hat{a}| \leq 1$, for $|\underline{z}| = \underline{1}$, whenever $\hat{a} \neq 0$. However, since $\hat{a}$ is scattering Schur, both $(\tilde{b}/\hat{a})\underline{z}^{-\underline{n}_a}$ and $(\tilde{a}/\hat{a})\underline{z}^{-\underline{n}_a}$ are holomorphic in $|\underline{z}| < \underline{1}$, and thus an application of lemma 2.2.3.11 yields that $|H_{12}| \leq 1$ and $|H_{22}| \leq 1$ for $|\underline{z}| < \underline{1}$. Similar arguments with (2.5.8a) yield $|H_{11}| \leq 1$ and $|H_{21}| \leq 1$, thus completing the proof of the present theorem.

<u>Definition 2.5.10:</u> An (NxN) rational matrix $Z = Z(\underline{z})$ is said to be discrete positive if each entry $Z_{ij}(\underline{z})$ is holomorphic in $|\underline{z}| < \underline{1}$, and $Z(\underline{z}) + Z^{*t}(\underline{z})$ is nonnegative definite for $|\underline{z}| < \underline{1}$.

<u>Lemma 2.5.11:</u> Let $Z = Z(\underline{z})$ be a discrete positive matrix in k-variables. If $Z(\underline{z})$ is regular at $\underline{z}_0$ with $|\underline{z}_0| \leq \underline{1}$ then $Z(\underline{z}_0) + (Z(\underline{z}_0))^{*t}$ is nonnegative definite.

<u>Proof:</u> If $|\underline{z}_0| < \underline{1}$ then the result is obvious in view of definition 2.5.10. If $|\underline{z}_0| \leq \underline{1}$ but $|\underline{z}_0| \nless \underline{1}$ then consider the neighbourhood $N_\varepsilon = \{\underline{z} : ||\underline{z} - \underline{z}_0|| < \varepsilon\}$[1] of $\underline{z}_0$ such that $Z(\underline{z})$ is holomorphic in $N_\varepsilon$. The existence of such a $N_\varepsilon$ follows from property 2.A5 in the appendix. Thus, in particular, each element of the matrices $Z(\underline{z})$ and $Z^*(\underline{z})$ are continuous functions of $\underline{z}$ in the connected set $N = N_\varepsilon \cap \{\underline{z} : |\underline{z}| < \underline{1}\}$. Consequently, for any constant vector $x$, $x(Z+Z^{*t})x^t$ is a continuous function of $\underline{z}$ in N. Furthermore, $x(Z+Z^{*t})x^t \geq 0$ for $\underline{z} \varepsilon N$, due to definition 2.5.10. It then follows by invoking the continuity of $x(Z+Z^{*t})x^t$ in N that $x(Z+Z^{*t})x^t \geq 0$ for $\underline{z} = \underline{z}_0$ and for arbitrary x.

The following result is a generalization of theorem 2.3.5 to multiports.

<u>Property 2.5.12:</u> Let $Z(\underline{z}) = [Z_{ij}(\underline{z})]$ be a discrete positive matrix. Then the denominators of each entry $Z_{ij}(\underline{z})$, when expressed in irreducible form, are immittance Schur polynomials.

<u>Proof:</u> The fact that the denominators of each entry of $Z_{ij}(\underline{z})$ is widest sense Schur obviously follows from definition 2.5.10. Let $Z'(z_1) = Z(z_1, \underline{z}_0')$, where $\underline{z}_0'$ is such that $|\underline{z}'_0| = \underline{1}$ and let $a = a(\underline{z})$ be the l.c.m of the denominator polynomials

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[1] $||.||$ denotes the Eucledean norm i.e., $||\underline{z}|| = \sum_{i=1}^{k} |z_i|^2$.

of the elements of the matrix $Z(\underline{z})$. Let $\nu$ be the multiplicity of a selfparaconjugate factor $c$ in the denominator of $Z_{ij}(\underline{z})$ in irreducible form. Then due to property 2.A4, $c'=c'(z_1)=c(z_1,\underline{z}_0')$ will be a factor of multiplicity $\nu$ in the denominator of $Z_{ij}'(z_1)=z_{ij}'(z_1,\underline{z}_0')$ in irreducible form for any choice of $\underline{z}'_0$ on $|\underline{z}'|=\underline{1}$ from a sequentially almost complete set $\Omega$ of (k-1) tuples of order (k-1). We note the polynomial $a'=a'(z_1)=a(z_1,\underline{z}'_0)$ cannot be identically zero for all $\underline{z}_0$ in $\Omega$, beacuse otherwise a would be identically zero due to property 2.A2 in the appendix. Consequently, there exists $\underline{z}_0 \in \Omega$ with $|\underline{z}_0|=\underline{1}$ such that $Z_1'(z_1)$ is well defined. By invoking lemma 2.5.11 it then follows that $Z'(z_1)+(Z'(z_1))_*^t$ is a discrete positive matrix function in the variable $z_1$ only. Since $c'$ is a selfparaconjugate factor in the denominator of $Z_{ij}'$ it immediately follows from known one-variable results that $\nu = 1$.

<u>Corollary 2.5.13:</u> If $Z=Z(\underline{z})$ is a discrete positive matrix then the : ast common denominator of all entries $Z_{ij}(\underline{z})$ of $Z(\underline{z})$ is an immittance Schur polynomial i.e., there exists an immittance Schur polynomial a such that aZ = polynomial matrix.

<u>Proof:</u> Follows from the fact that l.c.m. of a set of immittance Schur polynomials is also an immittance Schur polynomial (theorem 2.4.9c).

## 2.6. RELATIONSHIP BETWEEN MULTIDIMENSIONAL SCHUR POLYNOMIALS AND MULTIDIMENSIONAL HURWITZ POLYNOMIALS:

Let $g(\underline{p})$ and $a(\underline{z})$ be polynomials in the variables $\underline{p}$ and $\underline{z}$ respectively. Also let the polynomial $a(\underline{z})$ be obtained from $g(\underline{p})$ via the bilinear transformations $p_i = (1-z_i)/(1+z_i)$ for $i=1$ to $k$ as:

$$a(\underline{z}) = g\left(\frac{1-z_1}{1+z_1}, \frac{1-z_2}{1+z_2}, \ldots \frac{1-z_k}{1+z_k}\right) \prod_{i=1}^{k} (1+z_i)^{m_i} \qquad (2.6.1)$$

where, $m_i = \deg_i g$ for $i = 1$ to $k$.

Then the polynomial $a = a(\underline{z})$ has been called [7] the associate of the polynomial $g = g(\underline{p})$. On the otherhand if a and g are related as in (2.6.2) below then g is called [7] the associate of a.

$$g(\underline{p}) = a\left(\frac{1-p_1}{1+p_1}, \frac{1-p_2}{1+p_2}, \ldots \frac{1-p_k}{1+p_k}\right) \prod_{i=1}^{k} (1+p_i)^{n_i} \qquad (2.6.2)$$

where $\deg_i a = n_i$ for $i = 1$ to $k$.

General properties of associates of polynomials in $\underline{z}$ and in $\underline{p}$ variables and their interrelations are discussed extensively in [7]. We only note here that two polynomials may not be associates of each other. Consider, for exmple, $a = \prod_{1=1}^{k} (1+z_i)$ and $g = 2^k$.

In this context it has been shown in [7] that any scattering Schur polynomial is the associate of a scattering Hurwitz polynomial. Conversely, any scattering Hurwitz polynomial devoid of factors $(1+p_i)$ for any $i = 1$ to $k$, is also associate of some scattering Schur polynomial. Stated alternatively, the bileaner transformation along with its inverse transformation sets up a one-to-one correspondence between the scattering Schur polynomials and scattering Hurwitz polynomials devoid of factors of the type $(1+p_i)$. The scattering Hurwitz polynomials containing factors of the type $(1+p_i)$, when bilinearly transformed via $p_i = (1-z_i)/(1+z_i)$ also yield scattering Schur polynomials. This is in contrast to the situation discussed in [4], where it is shown that the associates of strict sense Hurwitz polynomials need not be strict sense Schur. In what follows answers to questions such these are sought in the context of other classes of multidimensional polynomials.

We first note that the associate of a discrete selfparaconjugate polynomial is necessarily selfparaconjugate i.e. if $\hat{a} = \gamma a$ then $g_* = \gamma g$, but the converse is not true i.e., the associate of selfparaconjugate polynomial is not necessarily discrete selfparaconjugate. However, it is true that if $g_* = \gamma g$ then $\hat{a} = \gamma . a \prod_{i=1}^{k} z_i^{r_i}$, where $r_i = \deg_i g - \deg_i a$ = multiplicity of the factor $(1+p_i)$ in g. However, if g is selfparaconjugate Schur then its irreducible factors must also be so, and therefore, g cannot have a factor $(1+p_i)$. Thus, if g is selfparaconjugate then its associate a is also discrete selfparaconjugate. Furthermore, the associate of a widest sense Hurwitz polynomial is widest sense Schur, and conversely the associate of a widest sense Schur polynomial is widest sense Hurwitz [7]. It, therefore, follows that there is a one-to-one correspondence under the bilinear transformation and its inverse between the selfparaconjugate Schur polynomials devoid of factors of the type $(1+z_i)$and the selfparaconjugate Hurwitz polynomials. Furthermore, selfparaconjugate Schur polynomials containing factor of the type $(1+z_i)$, under the action of the transformation $p_i = (1-z_i)/(1+z_i)$ i=1 to k necessarily yield selfparaconjugate Hurwitz polynomials. Since it is known [7, lemma 9] that if two polynomials are relatively prime their associates are also relatively prime the above comments also apply, in particular, to reactance Hurwitz polynomials. Furthermore, since the immittance Hurwitz (Schur) polynomials are products of reactance Hurwitz (Schur) polynomials and scattering Hurwitz (Schur) polynomials the following conclusions follow. The whole class of immittance Schur polynomials devoid of factors of the type $(1+z_i)$ can be identified as the class of associates of the whole class of immittance Hurwitz polynomials. On the other hand, the whole class of immittance Hurwitz polynomials devoid of factors $(1+p_i)$ for i = 1 to k can be identified as the class of associates of the class of immittance Schur polynomials. Furthermore, if a is any immittance Schur polynomial not containing $(1+z_i)$ for any i = 1 to k, as a factor then there exists an immittance Hurwitz polynomial g without the factor $(1+p_i)$ such that a and g are associates of each other. Conversely, for any immittance Hurwitz polynomial g devoid of the factor $(1+p_i)$ for all i = 1 to k there exists an immittance Schur polynomial a such that g and a are associates of each other. Thus, there is a one-to-one correspondence between the members of the class of immittance Schur polynomials devoid of factors of the type $(1+z_i)$ for i = 1 to k and the members of the class of immittance Hurwitz polynomials devoid of factors of the type $(1+p_i)$ for i = 1 to k.

The above discussion can be summarized in the form of theorem 2.6.1 for the purpose of which the following concise notation is adopted. The set of scattering Hurwitz (Schur), selfparaconjugate Hurwitz (Schur), reactance Hurwitz (Schur) and immittance Hurwitz (Schur) polynomials are denoted respectively by **SH** (**SS**), **PH** (**PS**), **RH** (**RS**) and **IH** (**IS**). Also, the subclass of a certain class of polynomials devoid of factors of the type $(1+p_i)$ or $(1+z_i)$ for all i, is to be denoted by adding the subscript 1+p or 1+z, as the case may be, to the symbol designating the corresponding class of polynomials. For example, the class of scattering Hurwitz polynomials devoid of factors of the type $1+p_i$, for all i will be denoted by $\mathbf{SH}_{1+p}$, whereas the class of selfparaconjugate Schur polynomials deviod of factors of the type is to be denoted by $\mathbf{PS}_{1+z}$. Furthermore, the notation A(C) will denote the the set of polynomials obtained by considering the associates of all polynomials belonging to a class C.

Theorem 2.6.1: The following set inclusion relations hold true.

(i) $A(\mathbf{SH}_{1+p}) = A(\mathbf{SH}) = \mathbf{SS}_{1+z} = \mathbf{SS}$; $A(\mathbf{SS}) = \mathbf{SH}_{1+p} \subset \mathbf{SH}$. The elements of $\mathbf{SH}_{1+p}$ and those of $\mathbf{SS}_{1+z}$ are in 1-1 correspondence.

(ii) $A(\mathbf{PH}) = \mathbf{PS}_{1+z} \subset \mathbf{PS}$; $A(\mathbf{PS}_{1+z}) = A(\mathbf{PS}) = \mathbf{PH} = \mathbf{PH}_{1+p}$. The elements of $\mathbf{PH}_{1+p}$ and $\mathbf{PS}_{1+z}$ are in 1-1 correspondence.

(iii) $A(\mathbf{RH}) = \mathbf{RS}_{1+z} \subset \mathbf{RS}$; $A(\mathbf{RS}_{1+z}) = A(\mathbf{RS}) = \mathbf{RH} = \mathbf{RH}_{1+p}$. The elements of $\mathbf{RH}_{1+p}$ and $\mathbf{RS}_{1+z}$ are in 1-1 correspondence.

(iv) $A(\mathbf{IH}_{1+p}) = A(\mathbf{IH}) = \mathbf{IS}_{1+z} \subset \mathbf{IS}$; $A(\mathbf{IS}_{1+z}) = A(\mathbf{IS}) = \mathbf{IH}_{1+p} \subset \mathbf{IH}$. The elements of $\mathbf{IH}_{1+p}$ and those of $\mathbf{IS}_{1+z}$ are in 1-1 correspondence.

## 2.7. SUMMARY:

Properties of widest sense Schur, selfparaconjugate Schur and scattering Schur polynomials previously not published in the literature are discussed. Elementary properties of discrete multidimensional positive functions are then studied. The reactance Schur and the immittance Schur polynomials occurring respectively as the numerators and denominators of discrete reactance functions and discrete positive functions are introduced for the first time. Various alternate characterizations of these polynomials are then suggested. The heirarchial ralationship between these different classes of Schur polynomials can be summarized diagramatically as is done, for example, in the case of the corresponding classes of Hurwitz polynomials in [2]. The role of these polynomials in scattering or immittance descriptions of discrete (k-D) passive multiports are investigated. In particular, a discrete (k-D) counterpart of Belevitch canonical form well known in classical network theory having potential applications in passive multidimensional digital filter design is derived. Relationship between the classes of multidimensional Schur polynomilas and the corresponding classes of multidimensional Hurwitz polynomials are also discussed within the present context, and are shown diagrammatically in fig 2.1.

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## APPENDICES

2.A. The proofs of the results stated below are ommitted for brevity [2].

<u>Definition 2.A1:</u> Let P be a set of k-tuples, $\underline{z} = (z_1, z_2, \ldots z_k)$, where all $z_i$ belong to a same number field $\Omega$ (e.g., either the field of real numbers or the field of complex numbers). We designate by $\Omega_i$, i = 1 to k, subsets of $\Omega$ and we say that $\Omega_i$ is almost complete if it comprises almost all elements of $\Omega$ ( i.e., all elements except at most finitely many). We then say that P is a <u>sequentially almost complete set</u> of order $m \geq 1$, with $m \leq k$, if there exists a permutation $i_1, i_2, \ldots i_k$ of the integers 1,2,...k such that all $z \in P$ can be generated in the following way. There exists an almost complete $\Omega_{i_1}$ such that any $z_{i_1} \in \Omega_{i_1}$ may be chosen. For any choice thus made, assuming $m \geq 2$, there there exists an almost complete $\Omega_{i_2}$ (possibly depending on the particular $z_{i_1} \in \Omega_{i_1}$ selected) such that any $z_{i_2} \in \Omega_{i_2}$ may be chosen. Again for any choice thus made, assuming $m \geq 3$, there exists an almost complete $\Omega_{i_3}$ ( possibly depending on the particular $z_{i_1}$ and $z_{i_2}$ selected) such that any $z_{i_3} \in \Omega_{i_3}$ may be chosen etc. If m = k this process is continued until we have reached $p_{i_k}$. If $m < k$, once we have reached $i_m$ there exists at least one (k-m) tuple $(z_{i_{m+1}}, \ldots z_{i_k})$ (possibly depending on the particular $z_{i_1}$ to $z_{i_m}$ selected) that may be chosen. Finally, we may extend the above definition to the situation m = 0 by saying that in this case P is not empty.

The set P is said to be <u>sequentially infinite</u> if in the above definition the term almost complete is replaced by infinite.

<u>Theorem 2.A2:</u> If a is a polynomial in k-variables such that the set of zeros of a comprises a sequentially infinite set of order k then a is identically equal to zero.

<u>Theorem 2.A3:</u> If and b are polynomials in k-variables then a and b have a proper common factor if and only if the set of common zeros that are common to a and b is a sequentially infinite set of order (k-1).

<u>Theorem 2.A4:</u> Let a nd b be two relatively prime polynomials. For any m such that $1 \leq m < k$ let us freeze m of the variables $z_i$, say for $i_1$ to $i_m$ at

corresponding values $z_{i0}$. Let $a_1$ and $b_1$ be the resulting poynomials in the remaining variables. Then there exists a sequentially almost complete set $\Omega_m$ of m-tuples of order m such thatfor $(z_{i_{10}}, z_{i_{20}}, \ldots z_{i_{k0}}) \varepsilon \Omega_m$, the polynomial $a_1$ and $b_1$ are still relatively prime. Furthermore, any ordering may be chosen for $\Omega_m$.

<u>Theorem 2.A5:</u> If $a(\underline{z})$ is a polynomial and $a(\underline{z}_0) \neq 0$ for some $\underline{z}_0$ then there exists $\varepsilon > 0$ such that $a(\underline{z}) \neq 0$ for all $\underline{z}$ in the neighbourhood $||\underline{z}-z_0|| < \varepsilon$, where $||.||$ denotes the Euclidean norm.

2.B. Various properties of the operations '$\tilde{\ }$' and '$\hat{\ }$', as defined in the introductory section of the present paper, are derived in this section.

<u>Property 2.B1:</u> If $a = b.c$ then $\tilde{a} = \tilde{b}\tilde{c}$ and $\hat{a} = \hat{b}.\hat{c}$, where b and c are polynomials.

<u>Property 2.B2:</u> If a is any polynomial then $\hat{a}$ cannot contain the factor $z_i$ for any i.

<u>Property 2.B3:</u> If the polynomial contains the factor $z_i$ with exact multiplicity $p_i$ then $(\deg_i a - \deg_i \hat{a}) = p_i$.

<u>Property 2.B4:</u> If the polynomial a does not have $z_i$ for any i as a factor, and $b = \hat{a}$ then $\hat{b} = a$.

<u>Property 2.B5:</u> If a is a widest sense Schur polynomial, and $b = \hat{a}$ then $\hat{b} = a$.

<u>Property 2.B6:</u> If the polynomial d is the greatest common factor between the polynomial a and its discrete paraconjugate $\hat{a}$ then d is a discrete selfparaconjugate polynomial.

<u>Property 2.B7:</u> Let $a' = a'(\underline{z}')$ be the polynomial obtained from by freezing in the polynomial $a = a(\underline{z})$ one of the variables, say $z_1$, at $z_1 = z_{10}$ on $|z_1| = 1$. Then $[\hat{a}]_{z_1=z_{10}} = z_{10}^{n_1} \hat{a}' \prod_{i=2}^{k} z_i^{p_i}$, where $p_i = \deg_i a - \deg_i a' \geq 0$ for all $i \neq 0$.

<u>Property 2.B8:</u> Let a be a discrete selfparaconjugate polynomial with $\hat{a} = \gamma a$ involving the variable $z_i$. If a is expressed as in (2.1.1) then for

$\nu=0,1,\ldots n_i$, (2.A1) holds true.

$$\gamma a_\nu = (\tilde{a}_{n_i-\nu})\Pi z_r^{n_r}, \text{ where the product extends over all } r\neq i \quad (2.A1)$$

<u>Property 2.B9:</u> If a is a polynomial and $\deg_i(a\pm\hat{a}) < \deg_i a$ for some i, then $(a\pm\hat{a})$ must contain the factor $z_i$.

<u>Property 2.B10:</u> If the polynomials a and b are such that $b = a\pm\hat{a}$ is widest sense Schur then $\hat{b} = \pm b$, i.e., b is either paraeven or paraodd respectively.

2.C. The polynomial $a_z$ associated with a is obtained from a via a formal algebraic operation as defined in section 2.1. It turns out that this operation plays the corresponding role of considering the derivative of a polynomial in the context of continuous systems. Related properties of $a_z$ are studied in the following.

<u>Property 2.C1:</u> If $f = gh$, where g and h are polynomials in z with their coefficients as polynomials over the field of complex numbers then $f_z = hg_z + gh_z$.

<u>Property 2.C2:</u> If f contains a factor g of multiplicity n ($n\geq 1$) then $f_z$ contains the factor g with exact multiplicity (n-1).

<u>Property 2.C3:</u> If the coefficients of the polynomial f belong to the field of complex numbers and $f\neq 0$ for $|z|<1$, then $\mathrm{Re}(f_z/f)>0$ for $|z|<1$.

# CHAPTER 3

## ON A GENERALIZED FACTORIZATION PROBLEM FOR THE SYNTHESIS OF QUARTER PLANE TYPE MULTIDIMENSIONAL DIGITAL FILTERS

### 3.1. INTRODUCTION:

Various synthesis schemes such as the Darlington synthesis scheme for synthesizing lossless transfer functions as a cascade interconnection of elementary lossless building blocks such as inductors, capacitors, gyrators etc. in the continuous domain are well known in classical network theory. The corresponding problem in the discrete domain, namely that of synthesizing a discrete lossless bounded (or positive) transfer function as a structurally passive interconnection of elementary lossless building blocks was first resolved via transformation from prototype problems in the continuous domain, and the resulting class of filter structures are now known as the wave digital filters [1], [17]. Recently, however, successful attempts to derive these and similar other discrete domain results without making explicit use of tools of classical network theory have been made. Notable among these are the orthogonal filters [2],[4],[14],[15] and the class of filters referred to as the lossless bounded real (LBR) filters described in [3], [13] and in related other publications.

In view of interest in the synthesis of multidimensional (k-D) wave digital filters, the problem of synthesis of k-D lossless two-port scattering transfer matrix via the bisection of a prescribed two-port into a cascade connection of two lossless two-port sections of smaller "degree" has been addressed in the continuous domain in [5]. Factorability of continuous domain two-port scattering matrices has also been studied recently [11] in the multidimensional context. An attempt to develop a complete and self consistent theory for the synthesis of k-D structurally passive quarter-plane causal type digital filters independent of the continuous domain methods have already been initiated in [8], [9] and [10] by discussing the discrete domain stability properties of a class of multidimensional polynomials. In the present paper the problem of synthesizing a k-D discrete quarter-plane causal type lossless two-port as a structurally passive interconnection of more elementary digital building blocks directly in the discrete domain is approached by the methods of factoring the chain matrix, the hybrid matrix and the transfer function matrix associated with a lossless two-port. By following recent results in [9] it can be shown that each of these matrices can be uniquely expressed by means of a set of

three polynomials in a form analogous to the Belevitch canonical form [16] of classical circuit theory. For the purpose of unified presentation of results, a matrix referred to as the (multidimensional) generalized lossless two-port matrix, which can be viewed as a generalization of the multidimensional chain matrix, the hybrid matrix and the transfer function matrix has been introduced. Interestingly, in 1-D this matrix can be categorized under the class of sigma lossless rational matrices considered, for example, in [12]. The problem of factorizing this generalized lossless two-port matrix into the product of two matrices of identical type can then be viewed as a problem of structurally passive synthesis of multidimensional two-ports. It must be noted that factorization of chain matrix, when feasible, yield networks having cascade structures as shown in Figure 3.1, whereas the factorization of hybrid matrix and transfer function matrix, when feasible, yield networks having the topological structure as shown in Figures 3.2 and 3.3.

Necessary and sufficient conditions for this generalized factorization problem so introduced to be solvable are obtained in the present paper via constructive techniques. As expected from our previous study of analogous problems in the continuous case [5],[11] it turns out that the factorization may not be feasible in a generic multidimensional ($k \geq 2$) situation. However, the impossibility of factorization of the chain matrix does not by any means rule out the feasibilty of factorization of the transfer function matrix. Exactly same comment also applies if the role of three types of matrices (i.e., chain, hybrid and transfer function) are permuted in any posssible manner (cf. Section 7). Furthermore, in order for the factorizations under consideration to yield computable digital filters the structures resulting from the factorization may not have any delay free loop. Apparently, this imposes a further constraint on the factorization not present in the corresponding continuous domain problems discussed in [5] and [11]. However, as shown in Section 3.5, this constrained problem can always be solved if and only if a solution to the unconstrained problem exists.

In the special case of 1-D, the criterion for factorability is always seen to be satisfied, thus guaranteeing the feasibility of factorization. Additionally, the factorization is seen to be nonunique. Our algorithm for computing these factors, however, enjoys two remarkable properties. First, it encompasses the

entire family of possible solutions. From this point of view it may be remarked that although synthesis in cascade type structures has previously been considered, for example, in [2],[14],[15] and [3], [13] our method, namely that of factoring the corresponding chain matrix, is somewhat more general. A second important property is that the factors can be computed essentially by solving a highly structured set of linear simultaneous equations, and thus can be potentially computed in a fast manner. On the other hand, although similar topological structures have been mentioned in [1] in the context of wave digital filters obtainable from analog prototypes, we are unaware of any previous work on discrete domain schemes for internally passive synthesis, which yield structures as a result of repeatative application of the decomposition indicated in Figures 3.2 and 3.3. Our discussion, even in the 1-D context, thus, yields a set of new algorithms for structurally passive synthesis of 1-D lossless digital filters previously not discussed in the literature.

Most multidimensional filtering tasks require the filter to have certain symmetries in their frequency response characteristics [7]. However, this demands that the two-port be either a symmetric or a (quasi) antimetric two-port. Motivated by potential applications of the results developed in the present paper to the design of multidimensional structurally passive digital filters, special attention to the synthesis of symmetric and (quasi) antimetric lossless two-ports have also been paid.

In section 3.2 a precise formulation of the problem along with some notation and terminology are introduced. In section 3.3 it is shown that the factorization problem so introduced is essentially algebraic in nature. An elementary step towards the general factorization problem is also taken here. In section 3.4 some properties of the fundamental equation which, in fact, is a linear version of the algebraic problem and is central to our study, are examined. Necessary and sufficient conditions for factorability and an algorithm for obtaining the factors, when they exist, are obatained in section 3.5. Section 3.6 discusses how the results so obtained yield new as well as known internally passive 1-D digital filter structures. In section 3.7 remarks are made on computational considerations of the algorithm for synthesis, examples are worked out to demonstrate the need for factorability of three

different kinds of matrices associated with discrete lossless two-ports, and the special cases of symmetric and (quasi) antimetric discrete two-ports are dealt with. Finally, conclusions are drawn in section 3.8.

## 3.2. NOTATIONS, TERMINOLOGY AND PROBLEM FORMULATION:

English capital letters are used to denote polynomials and rational functions in k-variables: $\underline{z} = (z_1, z_2, \ldots, z_k)$. The notation $\underline{z}^{\underline{n}}$ is used to denote the monomial $(z_1^{n_1} z_2^{n_2} \ldots z_k^{n_k})$ $\underline{n}$ being the k-tuple of nonnegative intergers $(n_1, n_2, \ldots, n_k)$. Also, $\tilde{A} = A^*(z_1^{*-1}, z_2^{*-1}, \ldots, z_k^{*-1})$ where * denotes complex conjugation. The notation $\deg_i A$ will be taken to mean the partial degree of the polynomial A in the variable $z_i$. Occassionally we shall also use the notation $\hat{A}$ to denote $\tilde{A}\underline{z}^{\underline{n}}$ with $\underline{n}=(n_1, n_2, \ldots n_k)$, where $n_i = \deg_i A$. Similarly for B and C etc. The notation $|\underline{z}| \leq 1$ denotes $|z_i| \leq 1$ for i=1 to k. Similar notations with $\leq$ replaced by $<$, $>$, $\geq$, $=$ etc. are also used.

The transfer function matrix $\Sigma$ associated with a k-D lossless two-port can be represented [9] as in (3.2.1). Note that (3.2.1) is slightly different although an equivalent version of the corresponding representation in [9]. Consequently, the chain matrix $\Theta$ and the hybrid matrix $\Gamma$ can also be represented as in (3.2.2) and (3.2.3) respectively. Also, note that a given $\Sigma$ can be uniquely represented as in (3.2.1) by requiring that $A(\underline{0})=1$. The representations (3.2.1), (3.2.2) and (3.2.3) can be regarded as canonic in this sense.

$$\Sigma = (1/A)\begin{bmatrix} B & \gamma\tilde{C}\underline{z}^{\underline{n}} \\ C & -\gamma\tilde{B}\underline{z}^{\underline{n}} \end{bmatrix} \qquad \Theta = (1/C)\begin{bmatrix} A & \gamma\tilde{B}\underline{z}^{\underline{n}} \\ B & \gamma\tilde{A}\underline{z}^{\underline{n}} \end{bmatrix} \qquad \Gamma = (1/B)\begin{bmatrix} A & -\gamma\tilde{C}\underline{z}^{\underline{n}} \\ C & -\gamma\tilde{A}\underline{z}^{\underline{n}} \end{bmatrix}$$

(3.2.1, 2.2, 2.3)

where (i) A is a scattering Schur polynomial [9] (3.2.4a)

(ii) $\gamma$ is a unimodular constant, i.e., $|\gamma| = 1$ (3.2.4b)

(iii) $A\tilde{A} = B\tilde{B} + C\tilde{C}$ (3.2.4c)

(iv) $\deg_i B \leq n_i$, $\deg_i C \leq n_i$ for all i=1 to k (3.2.4d)

Note that as a consequence of (3.2.4c) and (3.2.4d) we also have:

(v) $\deg_i A \leq n_i$ for all i=1 to k. (3.2.4e)

For the purpose of a unified presentation of the discussions that will follow,

matrices $\Theta$, $\Gamma$ and $\Sigma$ associated with a lossless two-port will be viewed as a matrix $\Phi$ associated with the lossless two-port as expressed in (3.2.5) while properties (P1) through (P4) hold true.

$$\Phi = (1/W)\begin{bmatrix} X & \sigma\tilde{Y}\underline{z}^{\underline{n}} \\ Y & \rho\sigma\tilde{X}\underline{z}^{\underline{n}} \end{bmatrix} \tag{3.2.5}$$

<u>Property 1 (P1)</u>: X, Y and W are polynomials; $\sigma$ = constant, $|\sigma|=1$, $|\rho|=\pm 1$.
<u>Property 2 (P2)</u>: $X\tilde{X} - \rho Y\tilde{Y} = W\tilde{W}$.
<u>Property 3 (P3)</u>: $\deg_i W \leq n_i$, $\deg_i X \leq n_i$ and $\deg_i Y \leq n_i$ for all i=1 to k.
<u>Property 4 (P4)</u>: X is scattering Schur when $\rho = 1$,
whereas W is scattering Schur when $\rho = -1$.

A (2 X 2) matrix such as the one in (3.2.5) will be said to be a <u>generalized lossless two-port matrix</u> if the above properties (P1) through (P4) hold true. Since it can be shown that $\Phi$ satisfy $\mathrm{diag}(1, -\rho) = \Phi.[\mathrm{diag}(1, -\rho)].\tilde{\Phi}$ and $\mathrm{diag}(1, -\rho) - \Phi.[\mathrm{diag}(1, -\rho)].\Phi^* \geq 0$ (i.e.,non-negative definite) in $|\underline{z}|<1$, where * denotes the Hermitian transpose, the one-dimensional counterpart of $\Phi$ thus, falls into the class of sigma-lossless transfer functions studied in [12]. Since factorability of the $\Phi$-matrices form the major topic discussed in the present paper, our results can also be viewed as a multidimensional generalization of the factorability of 1-D sigma-lossless transfer function elucidated in [12].

We note the following identification of the parameters of the matrix $\Phi$ in terms of the parameters of the chain matrix $\Theta$, the hybrid matrix $\Gamma$ or the transfer function matrix $\Sigma$.

(i) If $\Phi = \Theta$ then $W = C$, $X = A$, $Y = B$, $\sigma = \gamma$, $\rho = 1$
(ii) If $\Phi = \Gamma$ then $W = B$, $X = A$, $Y = C$, $\sigma = -\gamma$, $\rho = 1$
(iii) If $\Phi = \Sigma$ then $W = A$, $X = B$, $Y = C$, $\sigma = \gamma$, $\rho = -1$

An Advantage of the above formulation is that the problem of factoring the chain matrix, or the hybrid matrix or the transfer function matrix into a product of two non-trivial matrices of identical kind can be conveniently

formulated in a unified fashion as a single problem of factoring the generalized lossless two-port matrix $\Phi$ as $\Phi = \Phi'\Phi''$, where $\Phi'$ and $\Phi''$ are valid generalized lossless two-port matrices represented as in (3.2.6a) and (3.2.6b) with conditions analogous to (P1) through (P4) satisfied for $\Phi'$ as well as $\Phi''$.

$$\Phi' = (1/W')\begin{bmatrix} X' & \sigma'\tilde{Y}'\underline{z}^{\underline{n}'} \\ Y' & \rho\sigma'\tilde{X}'\underline{z}^{\underline{n}'} \end{bmatrix}; \quad \Phi'' = (1/W'')\begin{bmatrix} X'' & \sigma''\tilde{Y}''\underline{z}^{\underline{n}''} \\ Y'' & \rho\sigma''\tilde{X}''\underline{z}^{\underline{n}''} \end{bmatrix}; \qquad (3.2.6a, b)$$

Consider first two generalized lossless two-port matrices $\Phi'$, $\Phi''$ expressible respectively in terms of $X'$, $Y'$, $W'$, $\sigma'$ and $X''$, $Y''$, $W''$, $\sigma''$ which satisfy properties analogous to those satisfied by $X$, $Y$, $W$, $\sigma$ in $\Phi$ as in (P1) to (P4). Specifically, one obtains:

(i) $X'$, $Y'$, $W'$ and $X''$, $Y''$, $W''$ are polynomials;
$\sigma'$ and $\sigma''$ are unimodular constants. (3.2.7)

(ii) $X'\tilde{X}' - \rho Y'\tilde{Y}' = W'\tilde{W}'$, $X''\tilde{X}'' - \rho Y''\tilde{Y}'' = W''\tilde{W}''$ (3.2.8a, b)

(iii) $\deg_i W' \le n_i'$, $\deg_i W'' \le n_i''$ for all $i=1$ to $k$ and (3.2.9a)
$\deg_i Y' \le n_i'$, $\deg_i X' \le n_i'$, $\deg_i Y'' \le n_i''$, $\deg_i X'' \le n_i''$
for all $i=1$ to $k$ (3.2.9b)

(iv) $X'$ and $X''$ are scattering Schur when $\rho=1$,
whereas $W'$ and $W''$ are scattering Schur when $\rho=-1$. (3.2.10)

Then the following fact holds true.

Fact 3.2.1: If $\Phi'$ and $\Phi''$ are generalized lossless two-port matrices then $\Phi = \Phi'\Phi''$ is also a generalized lossless two-port matrix.

Proof: Given $\Phi'$ and $\Phi''$ define $X$, $Y$, $W$ and $\sigma$ and $\underline{n}=(n_1,n_2,\ldots n_k)$ as follows.

$$W = W'W'' \qquad (3.2.11)$$

$$X = X'X'' + \sigma'\tilde{Y}'Y''\underline{z}^{\underline{n}'}, \quad Y = Y'X'' + \rho\sigma'\tilde{X}'Y''\underline{z}^{\underline{n}'} \qquad (3.2.12a, b)$$

$$\sigma = \rho\sigma'\sigma'' \qquad (3.2.13)$$

$$n_i = n_i' + n_i'' \text{ for each } i=1 \text{ to } k \qquad (3.2.14)$$

It then follows from $\Phi=\Phi'\Phi''$ in a straightforward manner that $\Phi$ can be expressed in terms of X, Y, W and $\sigma$ as in (3.2.5). Clearly, (3.2.12), (3.2.7) and (3.2.9b) show that $\Phi$ satisfy (P1) and straightforward algebraic manipulations involving (3.2.12), (3.2.8), (3.2.11) and $\rho = \pm 1$ show that $\Phi$ satisfy (P2). Considering the degree restriction imposed by (3.2.9) and (3.2.14) on (3.2.11) and (3.2.12) likewise shows that (P3) is satisfied. Note that when $\rho = -1$ i.e., W' and W" are scattering Schur polynomials then $W = W'W''$ is clearly scattering Schur [9], and (P4) is thus obviously satisfied. This completes the proof that when $\rho=-1$, $\Phi=\Phi'\Phi''$ as expressed in (3.2.5) via (3.2.11) through (3.2.14) is indeed a generalized lossless two-port matrix. On the other hand, when $\rho = 1$ i.e., X' and X" are scattering Schur, X as in (3.2.12a) need not necessarily be scattering Schur, but can be shown to be immittance Schur [9].

However, to prove that $\Phi$ is a generalized lossless two-port matrix we consider the rational function $F = X/(X'X'')$. In view of (3.2.12a) we also have:

$$F = X/(X'X'') = 1 + \sigma'(\tilde{Y}'/X')(Y''/X'')\underline{z}^{\underline{n}'} \qquad (3.2.15)$$

It follows from (3.2.8) with $\rho=1$ that on $|\underline{z}| = 1$ we have $|\tilde{Y}'/X'| \leq 1$, whenever $X' \neq 0$ and $|Y''/X''| \leq 1$, whenever $X'' \neq 0$. Thus, Re $F \geq 0$ for $|\underline{z}| = 1$, whenever $X' \neq 0$ and $X'' \neq 0$. Next, since X' and X" are scattering Schur, by invoking Lemma 3b in [9], it follows that Re $F \geq 0$ for $|\underline{z}| < 1$ i.e., F is a discrete positive function. Consequently, the numerator polynomial of F, in irreducible rational form, is a immittance Schur polynomial [9]. Note that any possible factor common to X'X" and X must be scattering Schur, because X' and X" are so. Thus, X is immittance Schur and can, therefore, be expressed as the product of a scattering Schur factor $X_1$ and a reactance schur factor D [9]. Let $D_1$ be any irreducible (thus, reactance schur [9]) factor of D and note that there exists a sequentially almost complete set $\Omega$ of order (k-1) of unimodular complex numbers [9] such that $D_1 = X = 0$ for any $\underline{z}_0 \in \Omega$. Consequently, in view of (P2) we conclude that $Y = W = 0$ for all $\underline{z}_0 \in \Omega$. Since $D_1$, X, Y, W have a sequentially almost complete (and thus sequentially infinite [9]) set of common zeros of order (k-1) and $D_1$ is assumed irreducible, $D_1$ must be a factor of X, Y and W [6]. Since $D_1$ is any irreducible factor of D, we then have that $X = X_1D$, $Y = Y_1D$, $W = W_1D$, where $X_1$, $Y_1$, $W_1$ are polynomials. Since D is reactance Schur $\hat{D} = \sigma_D D$ for some unimodular constant $\sigma_D$. Clearly, then $\tilde{X}\underline{z}^{\underline{n}} = (D\sigma_D)(\tilde{X}_1\underline{z}^{\underline{m}})$ and

$\tilde{Y}\underline{z}^{\underline{n}} = (D\sigma_D)(\tilde{Y}_1\underline{z}^{\underline{m}})$, where $\underline{m}=(m_1,m_2,\ldots m_k)$ with $m_i=n_i-\deg_i D$ for all $i=1$ to $k$. Thus, after cancelling the common factor $D$ from the numerator and denominator of each entry of (3.2.5), $\Phi$ can be written as in (3.2.16), where $\sigma_1 = \sigma\sigma_D$.

$$\Phi = (1/W_1)\begin{bmatrix} X_1 & \sigma_1\tilde{Y}_1\underline{z}^{\underline{m}} \\ Y_1 & \rho\sigma_1\tilde{X}_1\underline{z}^{\underline{m}} \end{bmatrix} \tag{3.2.16}$$

Since $X_1$ is scattering Schur, Property (P4) is satisfied by the representation (3.2.16) for $\Phi$. It can be further shown via trivial algebraic manipulations that $X_1$, $Y_1$, $W_1$, $\sigma_1$ in (3.2.16) satisfy properties corresponding to (P1) to (P3) because $X$, $Y$, $W$, $\sigma$ has been shown to satisfy the same properties.

Note that in the case $\rho=1$ if $D$ is a nonconstant polynomial involving, say, $z_i$, then the two-port associated with $\Phi$ is degenerate in the sense that in (3.2.16) $m_i < n_i=n_i' + n_i''$. The main problem addressed in the present paper, however, is the converse problem of finding a non-degenerate factorization $\Phi = \Phi'\Phi''$ of a prescribed generalized lossless two-port matrix $\Phi$ into two factors of its own kind. More specifically, we have the following problem.

MAIN PROBLEM: Given a generalized lossless two-port matrix $\Phi$ as in (3.2.5), two constants $\sigma'$, $\sigma''$ such that $|\sigma'| = |\sigma''| = 1$ and $\sigma = \sigma'\sigma''$, and the polynomial factorization $W = W'W''$ along with two k-tuples of nonnegative intergers $\underline{n}'=(n_1', n_2',\ldots n_k')$ and $\underline{n}''=(n_1'', n_2'',\ldots n_k'')$ such that $\deg_i W' \leq n_i'$, $\deg_i W'' \leq n_i''$ and $n_i = n_i' + n_i''$ for all $i=1$ to $k$, we seek a factorization $\Phi = \Phi'\Phi''$, or equivalently, find polynomials $X'$, $Y'$, $X''$ and $Y''$ such that (3.2.12) along with (3.2.8) and (3.2.9) hold. Furthermore, if $\rho=1$ (or $\rho=-1$) then we require $X'$ and $X''$ (or $W'$ and $W''$) to be scattering Schur.

It proves to be convenient to introduce the following two definitions:

Definition 3.2.1: The pair of polynomial two-tuples $\{X', Y'\}$ and $\{X'', Y''\}$ is said to be a solution to the algebraic equation if (3.2.12) along with (3.2.8) and (3.2.9b) are satisfied.

Note that the restrictions that the polynomials $X'$ and $X''$ or $W'$ and $W''$ be scattering Schur polynomials are not imposed at all in the above definition.

Definition 3.2.2: A polynomial triple $\{X', Y', Y''\}$ is said to satisfy the fundamental equation if (3.2.17) along with (3.2.18) holds true.

$$YX' - XY' = \rho\sigma' Y'' W' \tilde{W}' \underline{z}^{\underline{n}'} \qquad (3.2.17)$$

$$\deg_i X' \le n_i', \ \deg_i Y' \le n_i', \ \deg_i Y'' \le n_i'' \quad \text{for all } i=1 \text{ to } k \qquad (3.2.18a,b,c)$$

Note that (3.2.17) is obtained by adding the product of $Y'$ and (3.2.12a) to the product of (3.2.12b) and $(-X')$ and subsequently by using (3.2.8a). Obviously, then any solution of the algebraic equation also satisfies the fundamental equation. However, the converse statement is false, consider e.g., $X' \equiv 0$, $Y' \equiv 0$, $Y'' \equiv 0$. Note further that the algebraic equations (3.2.12) along with (3.2.8) and (3.2.9b) constitute a highly constrained nonlinear problem. It is shown in Section 3.4 that due to the inherent structures underlying the problem under consideration, solutions to this nonlinear equations can be obtained from a certain subclass of solutions to the fundamental equation, which, in contrast, is clearly linear. Thus, solutions to the algebraic equation can be conveniently characterized in terms of the solutions of the fundamental equation.

### 3.3. SOLUTION TO THE ALGEBRAIC EQUATION:

Clearly, any solution to the problem of factorization of $\Phi = \Phi'\Phi''$ is also a solution to the algebraic equation. The converse statement is obvious if $\Phi$ is such that $\rho=-1$ i.e., $W$ is scattering Schur. The validity of the converse statement when $\rho=1$ i.e., when $X$ associated with $\Phi$ is scattering Schur is nontrivial but can be proved as follows in Theorem 3.3.1. Thus, the problem of factoring $\Phi$ reduces to that of solving a purely algebraic problem namely that of finding a solution to the algebaic equations.

<u>Theorem 3.3.1:</u> Let the pair of polynomial two tuples $\{X', Y'\}$ and $\{X'', Y''\}$ constitute a solution to the algebraic equation. If $\rho = 1$ (or $\rho = -1$) then the polynomials $X'$ and $X''$ (correspondingly, $W'$ and $W''$) are scattering Schur. Thus, any solution to the algebraic equations is a solution to the problem of factoring $\Phi = \Phi'\Phi''$.

<u>Proof:</u> The case when $\rho = -1$ trivially follows from scattering Schur property of $W = W'W''$. When $\rho = 1$ i.e., $X$ is scattering Schur, consider the rational function defined as:

$$F = (X'X'')/X \tag{3.3.1}$$

Furthermore, by adding the product of (3.2.12a) and $(\tilde{X}')$ to the product of (3.2.12b) and $(-\rho\tilde{Y}')$ and subsequently by using (3.2.8a), one obtains $X'' = (X\tilde{X}' - \rho Y\tilde{Y}')/(W'\tilde{W}')$. Substituting the last expression in (3.3.1) straightforward manipulation yields the following:

$$F = (X'\tilde{X}'/W'\tilde{W}')[1 - (\rho Y\tilde{Y}'/X\tilde{X}')] \tag{3.3.2}$$

It follows respectively that from (P2) and (3.2.8a) that on $|\underline{z}|=1$ we have $|Y/X| \leq 1$, whenever $X \neq 0$ and $|\tilde{Y}'/\tilde{X}'| \leq 1$, whenever $X' \neq 0$. An examination of (3.3.2) yields that $\mathrm{Re}F \geq 0$ for $|\underline{z}| = 1$, wherever $F$ is well defined. Thus, from Lemma 3b in [9] it follows that $F$ is a discrete positive function. Consequently, the numerator polynomial of $F$, in irreducible form, is an immittance Schur polynomial. Note that any possible factor common to $X'X''$ and $X$ must be

scattering Schur, because $X$ is so [9], and thus, $X'X''$ is widest sense Schur (more specifically, immittance Schur). Next, if for some $\underline{z}_0$ on the distinguished boundary $|\underline{z}| = 1$, we have $X'' = 0$, then from (3.2.8b) it follows that $Y'' = 0$, which in turn due to (3.2.12a) imply that $X = 0$. Consequently, if $X'' = 0$ for all $\underline{z}_0 \in \Omega$, where $\Omega$ is sequentially infinite set [6] of order (k-1) of unimodular complex numbers then $X = 0$ for all $\underline{z}_0 \in \Omega$. However, this is impossible if $X$ is scattering Schur [9]. Therefore, $X''$ cannot have a sequentially infinite set of zeros of order (k-1) on the distinguished boundary. The scattering Schur property of $X''$ is thus established in view of Theorem 9 in [9]. Similar arguments hold for $X'$.

Proposition 3.3.2: Any generalized lossless two-port matrix $\Phi$ can be decomposed as $\Phi = \Phi_1\Phi_o\Phi_2$, where $\Phi_1$, $\Phi_o$, $\Phi_2$ are valid generalized lossless two-port matrices such that $\Phi_1$, $\Phi_2$ are diagonal and the polynomials $X\tilde{X}\underline{z}^{\underline{n}}$ and $W\tilde{W}\underline{z}^{\underline{n}}$ associated with $\Phi_o$ are coprime.

Proof: Let $H = \gcd(W, X)$, where $W=HW_1$, $X=HX_1$. Since $W$ or $X$ is scattering Schur $H$ is also so. It then follows from property (P2) of (3.2.5) that $H$ is a factor of $Y\hat{Y}$. Let $H=H'H''$, where $H'$, $H''$ are factors of $Y$, $\hat{Y}$ respectively. Since $H''$ divides $\hat{Y}$, $\hat{H}''$ divides $Y$. Due to the scattering Schur property of $H''$ inherited from $H$, $H''$ and $\hat{H}''$ are coprime and thus $Y=(H''\hat{H}'')Y_1$ for some polynomial $Y_1$. A direct substitution of the last equation along with $W=HW_1$, $X=HX_1$ in $X\tilde{X} - \rho Y\tilde{Y} = W\tilde{W}$ yields (3.3.3a) in the following.

$$X_1\tilde{X}_1 - \rho Y_1\tilde{Y}_1 = W_1\tilde{W}_1, \quad X_c\tilde{X}_c - \rho Y_c\tilde{Y}_c = W_c\tilde{W}_c \qquad (3.3.3a,b)$$

Next, let $F = \gcd(\hat{X}_1, W_1)$, where $\hat{X}_1 = F\hat{X}_c$, $W_1 = FW_c$ and the monomial factors of of maximal degree in $X_c$ and $X_1$ are identical (note that this uniquely defines $X_c$ upto a constant multiplier). Clearly, $F$, being a factor of $\hat{X}_1$ cannot have a monomial factor and thus, from (3.3.3a), $F$ must divide $Y_1\hat{Y}_1$. Let $F=F'F''$, where $F'$ divides $Y_1$ and $F''$ divides $\hat{Y}_1$. The last requirement implies $\hat{F}''$ divides $Y_1$. If $\rho=-1$ then $F$ is scattering Schur since $W_1$ is also so. On the other hand, if $\rho=1$ since $\hat{X}_1=F\hat{X}_o$, then $\hat{F}$ is scattering Schur, because $X=HX_1$ is also scattering Schur. Thus, either both $F'$ and $F''$ or both $\hat{F}'$ and $\hat{F}''$ are scattering Schur. Consequently, $F'$ and $\hat{F}''$ are coprime, and $Y_1 = (F'\hat{F}'')Y_c$ for some polynomial $Y_c$. Next, if we define matrices $\Phi_f = \mathrm{Diag}(H''F', \hat{H}''F')$, $\Phi_r = \mathrm{Diag}(H'F'', H'\hat{F}'')$ and $\Phi_c$

as: $[\Phi_c]_{11}=X_c/W_c$, $[\Phi_c]_{21}=Y_c/W_c$, $[\Phi_c]_{12}=(\tilde{Y}_c\underline{z}^{\underline{m}})/W_c$, $[\Phi_c]_{22}=(\tilde{X}_c\underline{z}^{\underline{m}})/W_c$, where $\underline{m}=(m_1,m_2,\ldots m_k)$, $m_i=n_i-\deg_i(HF)$ it follows in a straightforward manner that $\Phi=\Phi_r\Phi_c\Phi_f$ and each of the matrices so defined is a generalized lossless two-port matrix with same $\rho$. In particular, (3.3.3b) holds true, where $m_i$ is at least as large as $\deg_i X_c$, $\deg_i Y_c$, $\deg_i W_c$ for each i=1 to k.

Clearly, $W_c$ is coprime with $\hat{X}_c$. On the otherhand, since it follows from the definition of $X_c$ that $X_1=FX_c$, $W_c$ and $X_c$ cannot have a common factor, because otherwise, due to $W_1=FW_c$, $W_1$ and $X_1$ would not be coprime. Consequently, $X_c\hat{X}_c$ and $W_c\hat{W}_c$ are coprime. If $m_i=\deg_i W_c$ for all i=1 to k then the last conclusion also implies that the polynomials $W_c\tilde{W}_c\underline{z}^{\underline{m}}$ and $X_c\tilde{X}_c\underline{z}^{\underline{m}}$ are coprime, the proof of the present theorem is complete. Otherwise, diagonal lossless two-port matrices from left and/or right of $\Phi_c$ needs to be further extracted to satisfy the coprimeness requirement.

For this purpose, note that, due to (3.3.3b), any monomial factor of $X_c$ present in $W_c\tilde{W}_c\underline{z}^{\underline{m}}$ must also be a factor of $Y_c\tilde{Y}_c\underline{z}^{\underline{m}}$. Let $S=S_1S_2$ be such a factor of maximal (total) degree, where the monomial $S_1$ divides $Y_c$, and the monomial $S_2$ divides $\tilde{Y}_c\underline{z}^{\underline{m}}$. Consider polynomials $X_{c'}=X_c/S$, $Y_{c'}=Y_c/S_1$, $W_{c'}=W_c$ and the integer k-tuple $\underline{m}'=(m_1',m_2',\ldots m_k')$ with $m_i'=m_i-\deg_i S$. Then clearly $X_{c'}\tilde{X}_{c'}-\rho Y_{c'}\tilde{Y}_{c'}=W_{c'}\tilde{W}_{c'}$ with $\deg_i X_{c'}$, $\deg_i Y_{c'}$ and $\deg_i W_{c'}$ upper bounded by $m_i'$ for all i=1 to k holds true. Thus, any monomial factor of $X_{c'}\underline{z}^{\underline{m}'}$ present in $W_{c'}\tilde{W}_{c'}\underline{z}^{\underline{m}'}$ must also be a factor of $Y_{c'}\tilde{Y}_{c'}\underline{z}^{\underline{m}'}$. Let $T=T_1T_2$ be such a factor of maximal (total) degree, where the monomial $T_1$ divides $Y_{c'}$, and the monomial $T_2$ divides $\tilde{Y}_{c'}\underline{z}^{\underline{m}}$. Next, consider $X_o=X_{c'}$, $Y_o=Y_{c'}/T$ and the integer k-tuple $\underline{n}_o=(n_{o1}, n_{o2},\ldots n_{ok})$, where $n_{oi}=m_i'-\deg_i T$. By letting $W_o=W_c$ it is then routinely verified that $\Phi_c=\mathrm{Diag}(S_2, T_1).\Phi_o.\mathrm{Diag}(S_1, T_2)$, where $\Phi_o$ is a generalized lossless two-port matrix described by $X_o$, $Y_o$, $W_o$, $\underline{n}_o$ and $\sigma$ as in (3.2.5) such that $W_o\tilde{W}_o\underline{z}^{\underline{n}_o}$ is coprime with $X_o\tilde{X}_o\underline{z}^{\underline{n}_o}$. The proof of the theorem is then completed by setting: $\Phi_1=\mathrm{Diag}(S_2, T_1).\Phi_f$ and $\Phi_2=\mathrm{Diag}(S_1, T_2).\Phi_r$.

## 3.4. PROPERTIES OF THE FUNDAMENTAL EQUATION.

In this section certain properties of the fundamental equation (3.2.17) crucial to the development are studied under the assumption that the prescribed generalized lossless two-port matrix $\Phi$ be such that the polynomials $X\tilde{X}\underline{z}^{\underline{n}}$ and $W\tilde{W}\underline{z}^{\underline{n}}$ are relatively prime. As shown in Proposition 3.3.2 no loss of generality is incurred due to this assumption.

<u>Lemma 3.4.1</u>: If the polynomial triple $\{X', Y', Y''\}$ is a solution to the fundamental equation then there exist a polynomial $X''$ given by (3.4.1) such that the polynomial triple $\{\rho\tilde{Y}'\underline{z}^{\underline{n}'}, \tilde{X}'\underline{z}^{\underline{n}'}, -\rho\sigma'^{*}X''\}$ is also a solution to the fundamental equation.

$$X'' = P/(\tilde{X}\underline{z}^{\underline{n}}) = -Q/(W'\tilde{W}'\underline{z}^{\underline{n}'}) \qquad (3.4.1a)$$

where

$$P = (\tilde{X}'W''\tilde{W}''\underline{z}^{\underline{n}} + \sigma'^{*}\tilde{Y}''\underline{z}^{\underline{n}''}Y),\; Q = Y(\rho\tilde{Y}'\underline{z}^{\underline{n}'}) - X(\tilde{X}'\underline{z}^{\underline{n}'}) \qquad (3.4.1b, c)$$

<u>Proof</u>: One obtains (3.4.2) by adding the product of tilde of (3.2.17) and $\rho Y$ to the product of (P2) and $X'$ and subsequently by using (3.2.11), (3.2.14) and trivial manipulations.

$$(\tilde{X}\underline{z}^{\underline{n}})Q = -(W'\tilde{W}'\underline{z}^{\underline{n}'})P \qquad (3.4.2)$$

Due to the upper bounds on the degrees of $X'$, $Y'$, $Y''$ imposed by (3.2.18) and $\deg_i W'' \leq n_i''$, it follows from (3.4.1b, c) that P and Q are polynomials. The fact that $X''$ in (3.4.1a) is a polynomial then follows from (3.4.2) in view of relative primeness of $\tilde{X}\underline{z}^{\underline{n}}$ with $W'\tilde{W}'\underline{z}^{\underline{n}'}$. Since $Q = Y(\rho\tilde{Y}'\underline{z}^{\underline{n}'}) - X(\tilde{X}'\underline{z}^{\underline{n}'}) = \rho\sigma'(-\rho\sigma'^{*}X'')W'\tilde{W}'\underline{z}^{\underline{n}'}$, the triple $\{\rho\tilde{Y}'\underline{z}^{\underline{n}'}, \tilde{X}'\underline{z}^{\underline{n}'}, -\sigma'^{*}X''\}$ satisfies the fundamental equation.

The fact that $\deg_i(\rho\tilde{Y}'\underline{z}^{\underline{n}'})$ and $\deg_i(\tilde{X}'\underline{z}^{\underline{n}'})$, for each i=1 to k, is upper bounded by $n_i'$ is obvious. In order to prove that $\deg_i X'' \leq n_i''$ we first note that it follows from (3.4.1b, c) and the upper bounds on the degrees of X, Y, X', Y', X'' and Y'' that for all i=1 to k we have:

$$\deg_i P \le n_i + n_i'', \ \deg_i Q \le n_i + n_i' \qquad (3.4.3a, b)$$

It is then necessary to distinguish between the following two cases.

(i) If $\rho=1$ i.e., X is scattering Schur then $\deg_i(\tilde{X}\underline{z}^{\underline{n}}) = n_i$ for all i=1 to k. It then follows from the first equality in (3.4.1a) and (3.4.3a) that $\deg_i X'' \le n_i''$ for all i=1 to k.

(ii) If $\rho=-1$ i.e., W, and thus W′, is scattering Schur we consider two sets of indices $I_1$, $I_2$ such that $i \in I_1$ if $\deg_i W' = n_i'$, whereas $i \in I_2$ if $\deg_i W' < n_i'$. If $i \in I_1$ then due to scattering Schur property of W′ we have $\deg_i(W'\tilde{W}'\underline{z}^{\underline{n}'}) = 2n_i'$. The desired result then follows from (3.4.3b) and second equality in (3.4.1a). On the other hand, if $i \in I_2$ then from (3.2.9a), (3.2.11) and (3.2.14) it follows that $\tilde{W}\underline{z}^{\underline{n}}$ must have a factor $z_i$, and thus, X does not have a factor $z_i$ because $\tilde{W}\underline{z}^{\underline{n}}$ and X are assumed to be coprime. Consequently, $\deg_i \tilde{X}\underline{z}^{\underline{n}} = n_i$. The result then follows from (3.4.3a) and first equality in (3.4.1a).

<u>Lemma 3.4.2</u>: If $\{X_1', Y_1', Y_1''\}$ and $\{X_2', Y_2', Y_2''\}$ are two polynomials triples satisfying the fundamental equation then the identity (3.4.4) holds and is equal to a constant.

$$N = \rho\sigma'(Y_1''X_2' - X_1'Y_2'')/X = (X_1'Y_2' - X_2'Y_1')/(W'\tilde{W}'\underline{z}^{\underline{n}'}) \qquad (3.4.4)$$

<u>Proof</u>: One obtains an equivalent form of (3.4.4) by adding the product of the fundamental equation for $\{X_1', Y_1', Y_1''\}$ and $X_2'$ to the product of the fundamental equation for $\{X_2', Y_2', Y_2''\}$ and $(-X_1')$. Since X is assumed coprime with $W'\tilde{W}'\underline{z}^{\underline{n}'}$ and $(X_1'Y_2' - X_2'Y_1')$ is a polynomial, it follows that X must divide $(Y_1''X_2' - X_1'Y_2'')$. Thus, $N = \rho\sigma'(Y_1''X_2' - X_1'Y_2'')/X$ in (3.4.4) is a polynomial. To prove that N is a constant, note that the following inequalities hold true for all i=1 to k.

$$\deg_i(Y_1''X_2' - X_1'Y_2'') \le n_i, \ \deg_i(X_1'Y_2' - X_2'Y_1') \le 2n_i' \qquad (3.4.5a, b)$$

Consider two sets of indices $I_1$, $I_2$ such that $i \in I_1$ if $\deg_i X = n_i$, whereas $i \in I_2$ if $\deg_i X < n_i$. If $i \in I_1$ then from (3.4.5a) and first equality (3.4.4) it follows that N does not involve $z_i$. If $i \in I_2$ then $\tilde{X}\underline{z}^{\underline{n}}$ has the a factor $z_i$.

Consequently, due to the assumed relative primeness of $X\tilde{X}\underline{z}^{\underline{n}}$ and $W\tilde{W}\underline{z}^{\underline{n}}$ it follows from (3.2.9a), (3.2.11) and (3.2.14) that neither $\tilde{W}'$ nor $\tilde{W}'\underline{z}^{\underline{n}'}$ may have the factor $z_i$, which in turn respectively imply that $\deg_i(\tilde{W}'\underline{z}^{\underline{n}'})=n_i'$ and $\deg_i W'=n_i'$. Therefore, due to (3.4.5b) and the second equality (3.4.3a) N may not involve $z_i$. Thus, $N$ = constant.

Lemma 4.3: If {X', Y', Y"} is a polynomial triple satisfying the fundamental equation then the expression given in (3.4.6) is equal to a real constant.

$$K = (X'\tilde{X}' - \rho Y'\tilde{Y}')/(W'\tilde{W}') = (\sigma'\tilde{Y}'\underline{z}^{\underline{n}'}Y'' + X'X'')/X \qquad (3.4.6)$$

Proof: Consider in view of Lemma 3.4.1 two solutions $X_1' = X'$, $Y_1' = Y'$, $Y_1'' = Y''$ and $X_2' = \rho\tilde{Y}'\underline{z}^{\underline{n}'}$, $Y_2' = \tilde{X}'\underline{z}^{\underline{n}'}$, $Y_2'' = -\rho\sigma'^{*}X''$ to the fundamental equation. It then immediately follows from Lemma 3.4.2 that K in (3.4.6) is a constant. Since for $|\underline{z}|=1$ we have $X'\tilde{X}'=|X'|^2$, $Y'\tilde{Y}'=|Y'|^2$ and $W'\tilde{W}'=|W'|^2$, K is a real constant.

Lemma 3.4.4: If the polynomial triple {X', Y', Y"} is a solution to the fundamental equation then there exists an X" as given by Lemma 3.4.1 such that $\{\alpha X' + \beta\rho\tilde{Y}'\underline{z}^{\underline{n}'},\ \alpha Y' + \beta\tilde{X}'\underline{z}^{\underline{n}'},\ \alpha Y'' - \beta\rho\sigma'^{*}X''\}$ is also a solution to the fundamental equation, where $\alpha$ and $\beta$ are arbitrary complex numbers.

Proof: Follows clearly from Lemma 3.4.1 and the fact that the fundamental equation is linear.

## 3.5. FACTORIZATION OF $\Phi$:

A solution $\{X', Y', Y''\}$ to the fundamental equation will be called <u>nonsingular</u> if $X'\tilde{X}' \neq \rho Y'\tilde{Y}'$.

<u>Theorem 3.5.1</u>: The problem of factorization of $\Phi$ admits a solution if and only if there exists a nonsingular solution $\{X', Y', Y''\}$ to the fundamental equation.

<u>Proof</u>: Necessity obviously follows from (3.2.8a) and that $W=W'W''\neq 0$. If $\{X_1', Y_1', Y_1''\}$ is a nonsingular solution to the fundamental equation then due to Lemma 3.4.4, $X' = \alpha X_1' + \beta\rho\tilde{Y}_1'\underline{z}^{n'}$, $Y' = \alpha Y_1' + \beta\tilde{X}_1'\underline{z}^{n'}$, $Y'' = \alpha Y_1'' - \beta\rho\sigma_1'^{*}X_1''$ is a solution to the fundamental equation. Straightforward manipulation then yields:

$$(X'\tilde{X}' - \rho Y'\tilde{Y}')/(W'\tilde{W}') = (|\alpha|^2 - \rho|\beta|^2)K_1 \qquad (3.5.1)$$

$$K_1 = (X_1'\tilde{X}_1' - \rho Y_1'\tilde{Y}_1')/(W'\tilde{W}') \qquad (3.5.2)$$

Since due to Lemma 3.4.3 and nonsingularity of $\{X_1', Y_1', Y_1''\}$, $K_1$ is a nonzero constant, if $\alpha$ and $\beta$ are chosen to satisfy $(|\alpha|^2 - \rho|\beta|^2) = K_1^{-1}$ we have $(X'\tilde{X}'-\rho Y'\tilde{Y}') = W'\tilde{W}'$. Furthermore, there exists $X''$ such that $(\rho\tilde{Y}'\underline{z}^{n'}, \tilde{X}'\underline{z}^{n'}, -\rho\sigma_1'^{*}X'')$, by the virtue of Lemma 3.4.1, satisfies the fundamental equation.

We next show that $X'$, $X''$, $Y'$ and $Y''$ so obtained constitute a solution to the algebraic equation. Equation (3.2.12b) is obtained by adding the product of (3.2.17) and $\tilde{X}'\underline{z}^{n'}$ to the product of second equality of (3.4.1a) via (3.4.1c) and $(-Y')$ and subsequently by using (3.2.8a). Likewise, (3.2.12a) is obtained by adding the product of (3.2.17) and $\rho\tilde{Y}'\underline{z}^{n'}$ to the product of second equality of (3.4.1a) via (3.4.1c) and $(-X')$ and subsequently by using (3.2.8a). Finally, we obtain (3.2.8b) by substitituting (3.2.12a) and (3.2.12b) in (P2) and then using (3.2.11) and (3.2.8a). Thus, the pair of two-tuples $\{X', Y'\}$ and $\{X'', Y''\}$ satisfies the algebraic equation and via Theorem 3.1 is a solution to the problems of factorization of $\Phi$.

Two polynomials triples $\{X_1', Y_1', Y_1''\}$ and $\{X_2', Y_2', Y_2''\}$ each satisfying the

fundamental equation will be said to be linearly dependent if there exists constants $\alpha$ and $\beta$ not simultaneously zero such that $\alpha X_1' + \beta X_2' \equiv \alpha Y_1' + \beta Y_2' \equiv \alpha Y_1'' + \beta Y_2'' \equiv 0$.

Theorem 3.5.2: The problem of factorization $\Phi$ admits a solution if and only if there exists two linearly independent polynomial triples $\{X_i', Y_i', Y_i''\}$, $i=1,2$ each of which satisfy the fundamental equation.

Proof: Necessity: Let the polynomials $X'$, $Y'$, $X''$ and $Y''$ constitute a solution to the factorization problem. Clearly, $\{X', Y', Y''\}$ is a solution to the fundamental equation. Due to Lemma 3.4.1, therefore, $\{\rho \tilde{Y}' \underline{z}^{\underline{n}'}, \tilde{X}' \underline{z}^{\underline{n}'}, -\rho\sigma'^{*} X''\}$ is also a solution to the fundamental equation. We claim that these two solutions are linearly independent, because otherwise there would exist constants $\beta_1$, $\beta_2$ not simultaneously zero, such that $\beta_1 X' + \beta_2 \rho \tilde{Y}' \underline{z}^{\underline{n}'} \equiv \beta_1 Y' + \beta_2 \tilde{X}' \underline{z}^{\underline{n}'} \equiv 0$. Thus, $(X'\tilde{X}' - \rho Y'\tilde{Y}') \equiv 0$, which in view of (3.2.8a), would imply that $W' \equiv 0$, i.e., due to (3.2.11) that $W \equiv 0$, which is impossible.

Sufficiency: If one of the solutions $\{X_i', Y_i', Y_i''\}$, $i=1,2$ is *nonsingular* then sufficiency follows from Theorem 3.5.1. If both solutions are singular then the triple $\{X', Y', Y''\}$ obtained as: $X' = pX_1' + qX_2'$, $Y' = pY_1' + qY_2'$, $Y'' = pY_1'' + qY_2''$, where p and q are complex numbers, satisfy the fundamental equation. Straightforward algebraic manipulation via the singularity of the triples $\{X_i', Y_i', Y_i''\}$, $i=1,2$ then yields:

$$(X'\tilde{X}' - \rho Y'\tilde{Y}')/(W'\tilde{W}') = pq^{*}L + p^{*}q\tilde{L},\ L = (X_1'\tilde{X}_2' - \rho Y_1'\tilde{Y}_2')/(W'\tilde{W}') \qquad (3.5.3a,b)$$

Since due to Lemma 3.4.1 $\{\rho \tilde{Y}_2' \underline{z}^{\underline{n}'}, \tilde{X}_2' \underline{z}^{\underline{n}'}, -\rho\sigma^{*} X_2''\}$ is also a solution, by invoking Lemma 3.4.2 on the triples $\{X_1', Y_1', Y_1''\}$ and $\{\rho \tilde{Y}_2' \underline{z}^{\underline{n}'}, \tilde{X}_2' \underline{z}^{\underline{n}'}, -\rho\sigma^{*} X_2''\}$ it follows that L in (3.5.3b) is a constant i.e., $L=\tilde{L}=L^{*}$. Thus, the right hand side of (3.5.3a) is $2\mathrm{Re}(pq^{*}L)$, which, if $L\neq 0$, can be made equal to 1 by proper choice of p and q. With p, q so chosen $\{X', Y', Y''\}$ would thus be a nonsingular solution to the fundamental equation, and by invoking Theorem 3.5.1, it then follows that a solution to the problem of factorization of $\Phi$ exists.

The proof of the present theorem is next completed by showing that $L\neq 0$. For this, consider the following cases.

(i) $\rho=-1$: Assume for contradiction that L = 0, which due to (3.5.3a) implies that $\tilde{X}'X' + \tilde{Y}'Y' = 0$ and thus, $|X'|^2 + |Y'|^2 = 0$ for $|\underline{z}|=1$. Consequently, $X' \equiv 0$, $Y' \equiv 0$, and via (3.2.17) $Y'' \equiv 0$, which contradicts linear independence of $\{X_i', Y_i', Y_i''\}$, i=1,2.

(ii) $\rho = 1$: Assume for contradiction that L = 0, which due to (3.5.3b) implies $\tilde{X}_1'X_2' = \tilde{Y}_1'Y_2'$. Since $\{X_1', Y_1', Y_1''\}$ is singular, we have $\tilde{X}_1'X_1' = \tilde{Y}_1'Y_1'$. The last two equations together imply $X_2'/X_1' = Y_2'/Y_1' = H_2/H_1$, where $H_1$ and $H_2$ are coprime polynomials. Clearly, there exists polynomials $X_o'$, $Y_o'$ such that

$$X_1' = H_1X_o', \qquad Y_1' = H_1Y_o', \qquad X_2' = H_2X_o', \qquad Y_2' = H_2Y_o' \qquad (3.5.4a,b,c,d)$$

Considering the fundamental equation for the triple $\{X_1', Y_1', Y_1''\}$ and $\{X_2', Y_2', Y_2''\}$, we obtain (3.5.5a,b) where $Y_o''$ is defined via (3.5.5c).

$$Y_1'' = H_1Y_o'', \qquad Y_2'' = H_2Y_o'', \qquad (YX_o' - XY_o') = \rho\sigma'\underline{z}^{\underline{n}'}\tilde{W}'W'Y_o'' \qquad (3.5.5a,b,c)$$

Clearly, $Y_o''$ is a polynomial, since otherwise its least denominator would divide both $H_1$ and $H_2$, i.e., $H_1$ and $H_2$ would not be coprime.

Furthermore, it follows from (3.5.4a), (3.5.4b) and (3.5.5a) that the degrees of the polynomials in the triple $\{X_o', Y_o', Y_o''\}$ cannot exceed the degrees of the corresponding polynomials in the triple $\{X_1', Y_1', Y_1''\}$. Thus, in view of (3.5.5c) $\{X_o', Y_o', Y_o''\}$, and consequently due to Lemma 3.4.1, $\{\rho\tilde{Y}_o'\underline{z}^{\underline{n}'}, \tilde{X}_o'\underline{z}^{\underline{n}'}, -\rho\sigma^{*\prime}X_o''\}$ is a solution to the fundamental equation for some $X_o''$. This last mentioned equation along with (3.5.4) and the fundamental equations for $\{\rho\tilde{Y}_1'\underline{z}^{\underline{n}'}, \tilde{X}_1'\underline{z}^{\underline{n}'}, -\rho\sigma^{*}X_1''\}$ (cf. Lemma 3.4.1) and $\{\rho\tilde{Y}_2'\underline{z}^{\underline{n}'}, \tilde{X}_2'\underline{z}^{\underline{n}'}, -\rho\sigma'^{*}X_2''\}$ yield:

$$X_1'' = \tilde{H}_1X_o'', \qquad X_2'' = \tilde{H}_2X_o'' \qquad (3.5.6a,b)$$

Next, if we define $F_o = X_o'X_o''/X$, then by eliminating $X_o''$ between $F_o$ and the fundamental equation for the triple $\{\rho\tilde{Y}_o'\underline{z}^{\underline{n}'}, \tilde{X}_o'\underline{z}^{\underline{n}'}, -\rho\sigma^{*}X_o''\}$ one obtains (3.5.7).

$$F_o = (\tilde{X}_o'X_o'/\tilde{W}'W')[1 - (\rho Y\tilde{Y}_o'/\tilde{X}_o'X)] \qquad (3.5.7)$$

From Property (P2) of $\Phi$ in (3.2.5) we have $|Y/X| \leq 1$ on $|\underline{z}|=1$, whenever $X \neq 0$. Since $\{X_1', Y_1', Y_1''\}$ has been assumed to be a singular solution to the fundamental equation, we have $X_1'\tilde{X}_1' = Y_1'\tilde{Y}_1'$, thus via (3.5.4a, b), $X_o'\tilde{X}_o' = Y_o'\tilde{Y}_o'$, implying that $|X_o'/Y_o'| = 1$ on $|\underline{z}|=1$, whenever $Y_o' \neq 0$. Then (3.5.7) yields that $\mathrm{Re}F_o \geq 0$ for $|\underline{z}|=1$, wherever $F_o$ is well defined. Using an argument analogous to that used in the proof of Theorem 3.3.1 it then follows (via discrete positive nature of $F_o$) that $X_o'$, $X_o''$ are widest sense Schur polynomials, and thus, cannot contain $z_i$ as a factor for any $i=1$ to $k$. Since in (3.5.6) $X_1''$ and $X_2''$ are polynomials, it then follows that $H_1$ and $H_2$ are constants. This latter conclusion violates, due to (3.5.4) and (3.5.5a, b), the linear independence of $\{X_1', Y_1', Y_1''\}$ and $\{X_2', Y_2', Y_2''\}$.

The above result can, in fact, be further sharpened as follows. Note that a corresponding strong result for the continuous case, although true, was not given in [5].

Theorem 3.5.3: The problem of factorization of $\Phi$ admits a solution if and only if there exists exactly two linearly independent polynomial triples $\{X_i', Y_i', Y_i''\}$, $i=1,2$ each of which satisfy the fundamental equation.

Proof: Due to Theorem 3.5.2, it is only needed to show that one can have at most two linearly independent solutions. Assume for contradiction that $\{X_i', Y_i', Y_i''\}$, $i=1$ to $3$ be three linearly independent solution to the fundamental equation and $X' = \alpha_1 X_1' + \alpha_2 X_2' + \alpha_3 X_3'$, $Y' = \alpha_1 Y_1' + \alpha_2 Y_2' + \alpha_3 Y_3'$, $Y'' = \alpha_1 Y_1'' + \alpha_2 Y_2'' + \alpha_3 Y_3''$. Then $\{X', Y', Y''\}$ and, due to lemma 3.4.1, $\{\rho \tilde{Y}' \underline{z}^{\underline{n}'}, \tilde{X}' \underline{z}^{\underline{n}'}, -\rho\sigma' {}_*X''\}$ is also a solution to the fundamental equation. From Lemma 3.4.3, K as defined in (3.5.8) is a constant.

$$K = (X'\tilde{X}' - \rho Y'\tilde{Y}')/(W'\tilde{W}') = (\sigma' \tilde{Y}' \underline{z}^{\underline{n}'} Y'' + X'X'')/X \qquad (3.5.8)$$

Consider next the following cases:

(i) $\rho=1$: Clearly, we may choose $\alpha_i$, $i=1$ to $3$ such that $X'(\underline{0}) = Y''(\underline{0}) = 0$. Since $X(\underline{0}) \neq 0$ due to the scattering Schur property of X, it follows from (3.5.8) that $K=0$ and thus, again from (3.5.8) that $X'\tilde{X}' = \rho Y'\tilde{Y}'$. Next, define $F = (X'X'')/X$.

From the second equality of (3.4.1a) and (3.4.1c), one obtains $X'' = (X\tilde{X}' - \rho\tilde{Y}\tilde{Y}')/(W'\tilde{W}')$. Eliminating $X''$ from the last two equations it follows that $F = (X'\tilde{X}'/W'\tilde{W}')[1 - (\tilde{Y}Y'/\tilde{X}X')]$. From Property (P2) of $\Phi$ and $X'\tilde{X}' = \rho Y'\tilde{Y}'$, one respectively obtains for $|\underline{z}| = 1$ that $|Y/X| \leq 1$, whenever $X \neq 0$ and $|Y'/X'| = 1$, whenever $X' \neq 0$. Using an argument similar to that used in proving Theorem 3.3.1, it is then possible to show that F is a discrete positive function, and finally, $X'X''$ is widest sense Schur, which is in contradiction with our construction that $X'(\underline{0})=0$.

(ii) $\rho=-1$: We claim that there exists $\underline{z}_0$ on $|\underline{z}| = 1$, such that $W'\tilde{W}' \neq 0$ [9] and $\alpha_i$, i=1 to 3 can be chosen such that $X'(\underline{z}_0) = Y'(\underline{z}_0) = 0$. It then follows from (3.5.8) that for $\underline{z} = \underline{z}_0$, $K = 0$, i.e., $|X'|^2+|Y'|^2 \equiv 0$ implying $X' \equiv 0$, $Y' \equiv 0$, and via (3.2.17), $Y'' \equiv 0$, which contradicts linear independence of $\{X_i', Y_i', Y_i''\}$, i=1 to 3.

To substantiate the claim we show that there exists $\underline{z}=\underline{z}_0$ on $|\underline{z}| = 1$ with $W'\tilde{W}' \neq 0$ such that $X_1'Y_2' - X_2'Y_1' \neq 0$ and thus, it is possible for any nonzero $\alpha_3$ to solve the following linear simultaneus equations for $\alpha_1$ and $\alpha_2$.

$$X'(\underline{z}_o) = \alpha_1 X_1'(\underline{z}_o) + \alpha_2 X_2'(\underline{z}_o) + \alpha_3 X_3'(\underline{z}_o) = 0 \tag{3.5.9a}$$

$$Y'(\underline{z}_o) = \alpha_1 Y_1'(\underline{z}_o) + \alpha_2 Y_2'(\underline{z}_o) + \alpha_3 Y_3'(\underline{z}_o) = 0 \tag{3.5.9b}$$

For this, consider a solution $X_1' = X_o'$, $Y_1' = Y_o'$, $X_1'' = X_o''$, $Y_1'' = Y_o''$ to the factorization problem. Then from the necessity part of proof of Theorem 3.5.2 it follows that $X_2' = \rho\tilde{Y}_o'\underline{z}^{\underline{n}'}$, $Y_2' = \tilde{X}_o'\underline{z}^{\underline{n}'}$, $Y_2'' = -\rho\sigma'^{*}X_o''$ is also a solution to the fundamental equation and that $\{X_i', Y_i', Y_i''\}$, i=1,2 are linearly independent. Next, for some $\underline{z}_0$ on $|\underline{z}| = 1$ if $X_1'Y_2'-X_2'Y_1' = 0$ then $|X_o'|^2 + |Y_o'|^2 = 0$ i.e., $X_o' = 0$, $Y_o' = 0$, and thus, due to the equation corresponding to (3.2.8a) satisfied by $X_o'$, $Y_o'$ we have $W' = \tilde{W}' = 0$. Since $W'$ is scattering Schur, there exists $\underline{z}_o$ on $|\underline{z}|=1$ with $W' \neq 0$ [9] and, consequently, with $X_1'Y_2' - X_2'Y_1' \neq 0$.

The fundamental equation (3.2.17), when considered as a set of linear simultaneous equations involving the coefficients of the polynomials $X'$, $Y'$, $Y''$ along with the upper bounds on their degrees, turns out to be overdetermined in general (except when k=1). More explicitly, we note that the unknown

polynomials $X'$, $Y'$ and $Y''$ contain a total of u unknown coefficients, whereas the total number of linear simultaneous equations can easily be found to be equal to e, u and e being as given in (3.5.10a,b) below.

$$u = \prod_{i=1}^{k} (n_i''+1) + 2 \prod_{i=1}^{k} (n_i'+1), \quad e = \prod_{i=1}^{k} (n_i'' + 2n_i' + 1) \qquad (3.5.10a,b)$$

Since for $k>1$ we have $e>u$ in a generic situation a solution to the problem of factoring $\Phi$ into two matrices of identical kind may not exist.

Delay free loop: In order for the digital network synthesized via the factorization of $\Phi$ to be 'computable' it may not contain delay free loops arising from interconnection of two sections. It is known [1] that this problem can always be circumvented, at least in the one-dimensional case, by incorporating digital equivalents of unit elements. The structures resulting from factorizations $\Theta = \Theta'\Theta''$, $\Gamma = \Gamma'\Gamma''$, $\Sigma = \Sigma'\Sigma''$ are shown in Figures 3.1, 3.2 and 3.3 respectively. An examination of directions of signal flows in Figure 3.3 shows that the topological structure arising from the factorization of $\Sigma$ as $\Sigma = \Sigma'\Sigma''$ cannot contain any delay free loop at the junction of the two-ports $\Sigma'$ and $\Sigma''$. On the otherhand, Figures 3.1 and 3.2 clearly show that the topological structures arising from the factorization of $\Theta$ as $\Theta = \Theta'\Theta''$ and $\Gamma$ as $\Gamma = \Gamma'\Gamma''$ may contain delay free loops unless special attention is paid to this issue (note that both of these cases correspond to the choice $\rho=1$). However, as shown in the following, delay free loops at the junction of the two-ports associated with $\Phi'$ and $\Phi''$ may always be avoided by extracting an appropriate constant (generalized) lossless two-port matrix from $\Phi''$ and subsequently combining it with $\Phi'$ (the obvious alternative of extracting a constant matrix from $\Phi'$ and combining it with $\Phi''$ also apply).

Fact 3.5.4: Any generalized lossless two-port matrix $\Phi''$ with $\rho=1$ can be factored into the product of two matrices $\Phi_f''$ and $\Phi_r''$ of the same type i.e., $\Phi''=\Phi_f''\Phi_r''$, where $\Phi_f''$ = constant, and $\Phi_r''$ is such that the Y-polynomial associated with it assumes a zero value for $\underline{z}=\underline{0}$.

Proof: Let $\kappa = Y''(\underline{0})/X''(\underline{0})$, where $X''$ and $Y''$ are corresponding polynomials associated with $\Phi''$, and $\Phi_f''$ be defined as follows.

$$\Phi_f'' = 1/\sqrt{(1 - |\kappa|^2)} \begin{bmatrix} 1 & \kappa \\ \kappa^* & 1 \end{bmatrix} \tag{3.5.11}$$

Since $\rho = 1$, it respectively follows from Properties (P2) and (P4) that $|Y''/X''| \leq 1$, wherever $X'' \neq 0$ on $|\underline{z}|=1$ and that $X''$ is scattering Schur. The last two properties, due to an extended multidimensional version of the maximum modulus theorem proved in [9], imply that either $Y''/X''$ = unimodular constant or $|Y''/X''| < 1$ for $|\underline{z}| < 1$. In the former case we would have, due to Property (P2) satisfied by $\Phi''$, that $W'' \equiv 0$, which is impossible. In the latter case, we have, in particular, $|\kappa| = |Y''(\underline{0})/X''(\underline{0})| < 1$. Thus, $\Phi_f''$ is a generalized lossless two-port matrix with $\rho=1$. Since it is easily verified that $(\Phi_f'')^{-1}$ also satisfies this property it follows from Fact 3.2.1 that $\Phi_r'' = (\Phi_f'')^{-1}\Phi''$ is also a generalized lossless two-port matrix. Finally, the fact that Y-polynomial associated with $\Phi_r''$, say $Y_r''$, satisfies $Y_r''(\underline{0}) = 0$ follows from $Y_r'' = (Y''-\kappa^*X'')/\sqrt{(1-|\kappa|^2)}$ (obtained by considering the (2,1) entry of the matrix equation $\Phi_r'' = (\Phi_f'')^{-1}\Phi''$) and $\kappa = Y''(\underline{0})/X''(\underline{0})$.

Next, if $\Phi$ is factorable as $\Phi = \Phi'\Phi''$ then due to Fact 5.4 we may also write $\Phi = \Phi_c'\Phi_r''$, where $\Phi_c' = \Phi'\Phi_f''$, due to Fact 3.2.1, is a generalized lossless two-port matrix. Further, it is trivially verified that if $\Phi'$ and $\Phi''$ satisfies the requirements imposed in the 'Main problem' of Section 3.2 then $\Phi_c'$ and $\Phi_f''$ also satisfies the same requirements. Thus, $\Phi = \Phi_c'.\Phi_f''$ is a valid solution to the factorization problem.

If $\Phi''$ is viewed as a chain matrix $\Theta''$, then (assuming that the operation of shifting the factor $\Phi_f''$ from $\Phi''$ into $\Phi'$ has been carried out) we have that $Y''(\underline{0}) = \Theta_{21}''(\underline{0}) = 0$ i.e., the corresponding transfer function matrix would in view of (3.2.1) satisfy $\Sigma_{11}''(\underline{0}) = 0$. Consequently, there would be no direct path from 'a' to 'b' via $\Phi'' = \Theta''$ in Figure 3.2. Similarly, if $\Phi''$ is viewed as a hybrid matrix $\Gamma''$ then the corresponding trasnfer function matrix would satisfy $\Sigma_{21}''(\underline{0}) = 0$, thus guaranteeing no direct path from 'a' to 'b' via $\Phi'' = \Gamma''$ in Figure 3.3. In either case, no delay free loop exists at the junction of the two two-ports. Note further that $\Phi''$ as in (3.5.10) correspond to the chain matrix or the hybrid matrix of the well known Gray-Markel section.

Furthermore, when $\rho=1$, $X''$ is scattering Schur and thus $X''(\underline{0}) \neq 0$. It then follows from (3.2.12b) that if $Y(\underline{0}) = 0$ and $Y''(\underline{0}) = 0$ then $Y'(\underline{0}) = 0$. This fact guarantees that the prescribed generalized lossless two-port matrix $\Phi$ can be successively factored into product of generalized lossless two-port matrices of progressively lower complexity in such a way that the Y-polynomial associated with each of the factors of $\Phi$ except possibly the one at the extreme left when $\Phi$ is interpreted as chain matrix $\Theta$ is equal to zero for $\underline{z}=\underline{0}$. Similar considerations apply when $\Phi$ is a hybrid matrix $\Gamma$. Absence of delay free loops at each junction of the constituent two-ports, when a given two-port is fragmented into an interconnection of more elementary two-ports via the method of factoring $\Phi$, is thus guaranteed.

The algorithm for factoring $\Phi$ can then be summarized as follows:

Step 1: If the prescribed $\Phi$ be such that associated $X\tilde{X}\underline{z}^{\underline{n}}$ and $W\tilde{W}\underline{z}^{\underline{n}}$ are coprime then proceed to Step 2. Otherwise, factor $\Phi=\Phi_1\Phi_o\Phi_2$ as described in the proof of Proposition 3.3.2. Replace $\Phi$ by $\Phi_o$.

Step 2: Find, if possible, two linearly independent solution $\{X_i', Y_i', Y_i''\}$, i=1, 2 to the fundamental equation (3.2.17). In the 1-D case such a solution always exist. Factorization of $\Phi$ is impossible if such solutions are nonexistent.

Step 3: If at least one of the two linearly independent solution is nonsingular i.e., $X_i'\tilde{X}_i' \neq \rho Y_i'\tilde{Y}_i'$ for any i then proceed to Step 4. Otherwise, proceed to Step Step 5.

Step 4: Assuming that $\{X_1', Y_1', Y_1''\}$ is a nonsingular solution, find $X_1''$ from the second equality (3.4.1a) and (3.4.1c) where $Y'$ and $X'$ are replaced by $Y_1'$ and $X_1'$ respectively. Also, find $K_1$ from (3.5.2) and $\alpha$, $\beta$ such that $|\alpha|^2 - \rho|\beta|^2 = 1/K_1$. Finally, form $X'= \alpha X_1' + \beta\rho\tilde{Y}_1'\underline{z}^{\underline{n}'}$, $Y' = \alpha Y_1' + \beta\tilde{X}_1'\underline{z}^{\underline{n}'}$, $Y''= \alpha Y_1'' - \beta\rho\sigma'^{*}X_1''$ and proceed to Step 6.

Step 5: Find the constant L as in (3.5.3b) and p, q such that $2\mathrm{Re}(pq^{*}L) = 1$. Form $X' = pX_1' + qX_2'$, $Y' = pY_1' + qY_2'$, $Y'' = pY_1'' + qY_2''$.

Step 6: Find $X''$ from (3.2.12a). Thus, $\{X', Y'\}$ and $\{X'', Y''\}$ i.e., $\Phi'$ and $\Phi''$ are

obtained.

<u>Step 7</u>: If $\rho=1$ then from X", Y" associated with $\Phi''$, find $\kappa=Y''(\underline{0})/X''(\underline{0})$, $\Phi''_f$ as in (3.5.11), $\Phi''_r = (\Phi''_f)^{-1}\Phi''$ and let $\Phi''_c = \Phi'\Phi''_f$. Thus, $\Phi = \Phi''_c\Phi''_r$ without delay free loop at the junction.

<u>Remark</u>: Since $K_1$ in (3.5.2) is a real constant it is possible to choose real values of $\alpha$ and $\beta$ such that the right hand side of (3.5.1) is equal to 1. If $\Phi$ is real (i.e., X, Y, W have real coefficients and $\sigma=\pm1$), then $X'_1$, $Y'_1$, $Y''_1$ as a solution to the fundamental equation, and thus, X', Y', Y" must also have real coefficients if $\alpha$ and $\beta$ are chosen to be real. Since this implies that the coefficients of X" are real, the factors $\Phi'$ and $\Phi''$ would then also be real.

### 3.6. ONE-DIMENSIONAL SYNTHESIS AS A SPECIAL CASE:

In the one-dimensional case i.e., if $k = 1$, a closer examination of (3.5.10a,b) reveals that $u - e = 2$, and, therefore, there are two more unknown coefficients than the number of linear equations in the set of linear simultaneous equations which determine the solution to the fundamental equation. Thus, there are (at least) two linearly independent solutions of the fundamental equation, and in view of Theorem 3.5.2, the problem of factorization of $\Phi$ always admits of a solution. Consequently, structurally passive synthesis for $\Phi$ is achieved by performing a sequence of further factorizations of $\Phi'$ and $\Phi''$ into the same kind of matrices of progressively lower complexity i.e., $n_1' < n_1$, $n_1'' < n_1$ (since $n_1' + n_1'' = n_1$), until a stage is reached when each of the resulting matrices cannot be factorized any further. This latter situation corresponds to the case that each of the two-ports resulting from the decomposition satisfy $n_1 = 1$, i.e., $\deg_1 C \leq 1$ and $\deg_1 B \leq 1$ and $\deg_1 A \leq 1$. However, if the prescribed $\Phi$ is such that X, Y, W have real coefficients, $\sigma = \pm 1$ and W has complex roots then it is necessary to allow two-port sections with $n_1 = 2$, $\deg_1 W = 2$ if realization involving only real multipliers are sought. In order to avoid delay free loops at the junction of the two-ports we further require that two-port sections satisfy $Y(0) = 0$ when $\rho = 1$ or equivalently, $B = 0$ when $\Phi = \Theta$ and $C = 0$ when $\Phi = \Gamma$ for $z_1 = 0$. Two-port sections of the above types will be called elementary sections and can in turn be realized in structures possibly other than those considered here by exploiting synthesis techniques as discussed, for example, in [4]. Thus, the following elementary sections are obtained.

An arbitrary lossless chain matrix $\Theta$ with $n_1 = 1$, $B(0) = 0$ can be synthesized by using the procedure described in [4] in a structure given in Figure 4, whereas an abitrary lossless chain matrix $\Theta$ with $n_1 = 2$, $B(0) = 0$ can, by following the same procedure, be synthesized in the structure of Figure 3.5.

A lossless hybrid matrix with $n_1 = 1$ and $n_1 = 2$ (assuming $Y(0) = C(0) = 0$) can respectively be realized by the same elementary sections as described in the Figures 3.4 and 3.5, but after a clockwise rotation of the corresponding diagrams by an angle of 90 degrees.

On the other hand, an arbitrary lossless transfer function matrix $\Sigma$ (in this case, we may not assume $Y(0)$ to be zero) with $n_1 = 1$ or $n_1 = 2$ can be realized by using the elementary sections described above only after a Gray -Markel section has been extracted from the corresponding chain matrix (or hybrid matrix) so as to effect a zero value for $B(0)$.

Thus, an arbitrary lossless two-port can indeed be synthesized as an interconnection of Gary-Markel sections and the sections depicted in Figures 3.4 and 3.5 only. Note that sections of Figure 3.4 and 3.5 were introduced by Dewilde and Deprettere in the context of cascade synthesis [19], and can be viewed as scaled versions of interconnections of wave digital filter adapters [18].

## 3.7. DISCUSSIONS AND ILLUSTRATIVE EXAMPLES:

The purpose of this section is many fold. First, we examine the structure of the fundamental equation in somewhat more detail to facilitate the method of solution both for the 1-D and the k-D case. Although it has been remarked in Section 3.5 that the $\Phi$-matrix is generically not factorable in multidimensions, the possibility of synthesis for special classes of $\Phi$-matrices may not be ruled out. Furthermore, in k-D, nonfactorability of any one of the three matrices, $\Sigma$, $\Theta$, or $\Gamma$ associated with a lossless two-port does not rule out the factorability of other two matrices. This fact is next substantiated via examples, thus justifying the need to study factorization of all three types of matrices (in a unified manner). Finally, in practice, all multidimensional frequency filtering problems require some form of symmetry in the k-D frequency response, and it is known that such symmetries dictate that the two-port be either symmetric or (quasi) antimetric in the sense of classical network theory (to be made precise later in this section). This is indeed the case, for example, in the design of 2-D fan [20] and k-D circularly symmetric [21] wave digital filters based on transformations from analog prototypes. Therefore, the factorabilty of the $\Sigma$-matrix associated with these subclasses of discrete lossless two-ports is also undertaken in the present section.

A. Computational considerations:

For the purpose of the present discussion, the following notations will be adopted.

$$X(\underline{z})=\Sigma\, P_i(\underline{z}')z_1^i, \qquad Y(\underline{z})=\Sigma\, Q_i(\underline{z}')z_1^i, \qquad \rho\sigma W'\tilde{W}'\underline{z}^{\underline{n}'}=\Sigma\, R_i(\underline{z}')z_1^i \quad (3.7.1\ a,b,c)$$

$$X'(\underline{z})=\Sigma\, P_i'(\underline{z}')z_1^i, \quad Y'(\underline{z})=\Sigma\, Q_i'(\underline{z}')z_1^i, \qquad Y''(\underline{z})=-\Sigma\, R_i'(\underline{z}')z_1^i \quad (3.7.2\ a,b,c)$$

where $\underline{z}'$ is the k-tuple of integers $(z_2, z_3, \ldots z_k)$ and $P_i=P_i(\underline{z}')$, $Q_i=Q_i(\underline{z}')$, $R_i=R_i(\underline{z}')$ and $P_i'=P_i'(\underline{z}')$, $Q_i'=Q_i'(\underline{z}')$, $R_i'=R_i'(\underline{z}')$ are polynomials in $\underline{z}'$. Then the fundamental equation (3.2.17) can be written in the form of (3.7.3):

$$V(\underline{z}')T(\underline{z}')=0, \qquad T=T(\underline{z}')=[T_1^t \mid T_2^t \mid T_3^t]^t \quad (3.7.3\ a,b)$$

$$V(\underline{z}') = [Q'_{n_1},\ldots Q'_0 \mid -P'_{n_1},\ldots -P'_0 \mid R''_{n''_1},\ldots R''_0] \qquad (3.7.3\text{ c})$$

where the superscript 't' denotes matrix transposition, $T_1$ is a $(n'_1+1)\mathrm{x}(n_1+n'_1+1)$ lower shift matrix whose first row is the (k-1) variable polynomial matrix: $[P_{n_1},\ldots,P_0, 0,\ldots 0]$ and subsequent rows of which are obtained by shifting the previous rows by one step towards the right. The $(n'_1+1)\mathrm{x}(n_1+n'_1+1)$ matrix $T_2$ and the $(n''_1+1)\mathrm{x}(n_1+n''_1+1)$ matrix $T_3$ are similarly obtained from the polynomial row vectors: $[Q_{n_1},\ldots Q_0, 0,\ldots 0]$ and $[R_{2n_1},\ldots R_0, 0,\ldots 0]$ respectively. For a given $T(\underline{z}')^1$ a solution $V(\underline{z}')$ to (3.7.3a) corresponds to a solution to the fundamental equation if in (3.7.3c) the following degree restrictions for all $\nu$ are satisfied for each i=2 to k.

$$\deg_i Q'_\nu \le n'_i,\ \deg_i P'_\nu \le n'_i,\ \deg_i R''_\nu \le n''_i \qquad (3.7.4)$$

As remarked earlier, if $k \geq 1$ for a given $T(\underline{z})$ a solution $V(\underline{z})$ satisfying (3.7.3a) and (3.7.4) may not, in general, exist. However, the following approach may be adopted in attempting a viable solution. First, find the Hermite reduced form $H(\underline{z}')$ of $T(\underline{z}')$ via the pseudo-division algorithm as described, for example, in [22], [23] i.e., find a unimodular matrix $U(\underline{z}')$ such that $U(\underline{z}')T(\underline{z}') = H(\underline{z}')$ is in Hermite form. Since $T(\underline{z}')$, and thus, $H(\underline{z}')$ has $(n_1+n'_1+1)$ columns, but $(n_1+n_1+3)$ rows, the last $\kappa$ rows ($\kappa \geq 2$) of $H(\underline{z}')$ are identically zero. Consequently, each of the last $\kappa$ rows of $U(\underline{z}')$ belong to the left null space of $T(\underline{z}')$. However, in order forany vector belonging to this space to correspond to a solution of the fundamental equation, (3.7.4) must be satisfied, which, generically, may fail. To elaborate further on this it may be remarked that (proof ommitted for brevity) in the special case of 2-D i.e., when k=2 and if $n'_2=n''_2$ a necessary and sufficient condition for the existence of two linearly independent solutions to the funamental equation i.e., that of factorability of $\Phi$ is that the dimensionality of left null space of $T(\underline{z}')=T(z_2)$ be exactly two and the two left Kronecker indices [23] of the polynomial matrix $T(\underline{z}')=T(z_2)$ be each equal to $n'_2=n''_2$. We have so far been unable to establish an analogous characterizaion of factorability when $n'_2 \neq n''_2$.

In the 1-D case (k=1), however, both $V=V(\underline{z}')$ and $T=T(\underline{z}')$ in equation (3.7.3a) are constant matrices. Furthermore, since it is known that the lower shift

matrices $T_1$, $T_2$ and $T_3$ are closely related to Toeplitz as well as resultant-like matrices, the linear simultaneous equation (3.7.3a) can be potentially solved by exploiting recently developed fast algorithms for solving such equations [24]. Furthermore, it can be shown by pursuing the proof technique for Theorem 3.5.3 that in obtaining two linearly independent solutions to the fundamental equation, one of the zeros of the polynomial $Y''(z_1)$ may be chosen arbitrarily. Once such a choice is made, the solution to the fundamental equation becomes essentially unique except for a constant scale factor multiplying each of the polynomials $\{X', Y', Y''\}$ in the solution. It can be shown that each of these three polynomials in this solution can in turn be expressed via closed form determinantal formulas as discussed in [25]. From a computational standpoint this latter method, as opposed to the Toeplitz-like method mentioned above may not, however, be the most inexpensive when the integers $n_1'$, $n_1''$, and thus, n are large.

B. Examples on factorability of $\Sigma$, $\Theta$ and $\Gamma$:

We next illustrate by three examples that in multidimensions by viewing the $\Phi$-matrix as three different types of matrices associated with a two-port, namely the transfer function matrix $\Sigma$, the chain matrix $\Theta$ and the hybrid matrix $\Gamma$, the synthesis of a larger class of discrete lossless two-ports can be attempted than by considering the factorization of a matrices of only one of the above kinds.

(I) Consider a discrete lossless two-port described by $A = PQ$, $B$, and $C = 2RS$, $\underline{n} = (3,3)$ and $\sigma=-1$ as in (3.2.1), where

$$P=z_1z_2^2-2z_2z_1+z_1+4,\quad Q=z_1^2z_2-2z_1z_2+z_2+4,$$

$$B=4z_1^3z_2^3-4z_1^2z_2^3-5z_1^3z_2^2-6z_1^2z_2^2+7z_1z_2^2-2z_1^3z_2+4z_1^2z_2-6z_1z_2-4z_2-z_1^3-2z_1^2-5z_1+4,$$

$$R=z_1z_2-1,\quad S=z_1z_2^2-z_2^2+z_1^2z_2+4z_1z_2-z_2-z_1^2-z_1-2$$

In attempting a nondegenerate factorization of the corresponding $\Theta$-matrix ($\rho=1$) into nonconstant $\Theta'$ and $\Theta''$ we encounter the following distinct possibilities: (i) $\underline{n}'=(2,2)$, $\underline{n}''=(1,1)$, $Y'= C'= S$, $Y''= C''= 2R$ (ii) $\underline{n}'=(1,1)$, $\underline{n}''=(2,2)$, $Y'= C'$ =

R, $Y''= C'' = 2S$. Neither in case (i) nor in case (ii) we have two independent solutions to the fundamental equation (3.2.17). Via Theorem 3.5.3, we thus conclude that $\Theta$ cannot be factored as $\Theta'\Theta''$. However, the corresponding $\Sigma$ can factored as $\Sigma = \Sigma'\Sigma''$ where $\Sigma'$ and $\Sigma''$ are described as (clearly, $\rho=\rho'=\rho''=-1$ in this case):

$$W'= A' = P,\ X'= B'= 2z_1z_2^2-2z_1z_2-2z_2+2,\ Y'= C'= z_2^2+2z_2+1,\ \sigma'=-1$$

$$W''= A'' = Q,\ X''= B''= 2z_1^2z_2-2z_1z_2-2z_1+2,\ Y''= C''= z_1^2+2z_1+1,\ \sigma''=-1$$

Since the above $\Theta$ can also be viewed as a hybrid matrix $\Gamma$ (with slight modifiation in the sign of $\sigma$), the example also demonstrates that there exists discrete lossless two-ports for which the associated transfer function matrix can be factored but the associated hybrid matrix may not be factorable.

(II) Consider the discrete lossless two-port given by A, $B = 2PQ$, $C = RS$, $\underline{n}=(2,2)$, $\sigma=1$, where

$$A=3-z_1^2-z_2^2-z_1^2z_2^2,\ P=1-z_1z_2,\ Q=1+z_1z_2,\ R=z_1z_2+z_2+z_1+1,\ S=z_1z_2-z_2-z_1+1$$

An attempt to factor $\Gamma$ into nondegenerate nonconstant factors $\Gamma'$ and $\Gamma''$ gives rise to the following two distinct possibilities with $\underline{n}'=\underline{n}''=(1,1)$ in both cases: (i) $Y'= B'= 2P$, $Y''= B''= Q$ (ii) $Y'= B'= Q$, $Y''= B''= 2P$. In neither of the above two cases the fundamental equation is found to have two linearly independent solutions, thus proving, in view of Theorem 3.5.3, the impossibility of the intended factorizaion. However, the corresponding $\Theta$-matrix can be factored as $\Theta=\Theta'\Theta''$, where $\Theta'$ and $\Theta''$ are described withy $\rho=\rho'=\rho''=1$ as:

$$X'= A'=(z_1z_2+2z_1+2z_2+3)/\sqrt{3},\ Y'= B'=(z_1+z_2+2)/\sqrt{3},\ W'= C'= R,\ \ \sigma'=1$$

$$X''= A''=(z_1z_2-2z_1-2z_2+3)/\sqrt{3},\ Y''= B''=(-2z_1z_2+z_2+z_1)/\sqrt{3},\ W''= C''= S,\ \sigma''=1$$

Note that $Y''(\underline{0})= B''(\underline{0}) = 0$; thus, there is no delay free loop at the junction of the discrete two-ports. By interchanging the roles of B-polynomial and the C-polynomial in the above example we can similarly demonstrate the existence of a discrete lossless two-port for which the $\Gamma$-matrix can be factored as $\Gamma=\Gamma'\Gamma''$,

but the associated $\Theta$ is not factorable as $\Theta=\Theta'\Theta''$.

(III) Consider next a discrete lossless two-port described by A, B, C=RS, $\underline{n}=(2,2)$, $\sigma=1$, where

$$A=-(7+3z_1-2z_2-2z_1z_2-z_2^2+3z_1z_2^2),\ B=z_1^2z_2^2-3z_1z_2^2+z_1^2+z_1-2z_1^2z_2-2z_1z_2-4,$$

$$R=z_1+1,\ S=2z_1z_2^2-2z_1z_2-2z_2+2$$

A detailed examination of the degrees of the polynomials A, B and C reveals that the only possible way of factoring the transfer function matrix $\Sigma$ associated with the two-port is to attempt either (i) $\underline{n}'=(1,0)$, $\underline{n}''=(1,2)$, $A'=1$, $A''=A$ or (ii) $\underline{n}'=(1,2)$, $\underline{n}''=(1,0)$, $A'=A$, $A''=1$. In both cases, however, the fundamental equation (3.2.17) fails to yield two linearly independent solutions. Thus, $\Sigma$-matrix associated with the discrete two-port under consideration cannot be factored. On the otherhand, the associated chain matrix $\Theta$ can be factored as $\Theta=\Theta'\Theta''$, where $\Theta'$ and $\Theta''$ are described as:

$$\sigma''=1,\ \sigma'=1,\ W'=C'=R,\ W''=C''=S,\ X'=A'=-(z_1+7)/\sqrt{15},\ Y'=B'=(2z_1-4)/\sqrt{15},$$

$$X''=A''=(4z_1z_2^2-8z_1z_2+4z_1-z_2^2-2z_2+15)/\sqrt{15},\ Y''=B''=(-z_1z_2^2+4z_2^2+2z_1z_2+8z_2-z_1)/\sqrt{15}$$

Note again that $Y''(\underline{0})=B''(\underline{0})$ ensures that there is no delay free loop at the junction. By viewing the chain matrix as a hybrid matrix the same example with minor modofications can be used to show the existence of a discrete lossless two-port for which the transfer function matrix $\Sigma$ cannot be factored, although it is possible to factor the associated hybrid matrix $\Gamma$.

C. Symmetric and (quasi) antimetric two-ports:

A discrete lossless two-port will be called symmetric or (quasi) antimetric if (3.7.5) (or (3.7.5')) holds true. Note that the former case corresponds to fan type symmetry [20], whereas the latter case correponds to circular symmetry [21] in frequency response.

$$C=\sigma\tilde{C}\underline{z}^{\underline{n}},\quad B=-\sigma\tilde{B}\underline{z}^{\underline{n}},\qquad (C=-\sigma\tilde{C}\underline{z}^{\underline{n}},\ B=-\sigma\tilde{B}\underline{z}^{\underline{n}})\qquad (3.7.5)\ ((3.7.5'))$$

Let the rational functions $L_1$ and $L_2$ be defined as in (3.7.6) (or (3.7.6')) for symmetric (or (quasi) antimetric) filters.

$$L_1=(B+C)/A \quad (\text{or } (B+jC)/A\ ) \qquad (3.7.6a)\ ((3.7.6'a))$$

$$L_2=(B-C)/A \quad (\text{or } (B-jC)/A\ ) \qquad (3.7.6b)\ ((3.7.6'b))$$

Then it clearly follows from (3.2.4c) that $L_1\tilde{L}_1=L_2\tilde{L}_2=1$, and consequently, $|L_1|=|L_2|=1$ for $|\underline{z}|=1$, wherever $L_1$ or $L_2$ are well defined. Since both $L_1$ and $L_2$ have scattering Schur denominators [9], it follows that $L_1$ and $L_2$ are both multidimensional discrete all-pass functions. By making use of (3.7.6) (or (3.7.6')), $\Sigma$ as in (3.2.1) can be expressed as in (3.7.7a) and (3.7.7b) in the symmetric case and as in (3.7.7'a) and (3.7.7'b) in the (quasi) antimetric case.

$$\Sigma_{11}=\Sigma_{22}=(L_1+L_2)/2, \quad \Sigma_{12}=\Sigma_{21}=(L_1-L_2)/2 \qquad (3.7.7\ a,b)$$

$$\Sigma_{11}=\Sigma_{22}=(L_1+L_2)/2, \quad \Sigma_{21}=-\Sigma_{12}=j(L_2-L_1)/2 \qquad (3.7.7'\ a,b)$$

Conversly, for any two discrete all-pass functions $L_1$ and $L_2$ the matrix $\Sigma$ obtained by using (3.7.7) (or(3.7.7')) is a valid transfer function matrix of a discrete lossless two-port. Thus, any multidimensional discrete lossless two-port can be equivalently described by means of two multidimensional all-pass functions $L_1$, $L_2$. We then have the following important result.

Theorem 3.7.1: Let $\{L_1, L_2\}$, $\{L_1', L_2'\}$, $\{L_1'', L_2''\}$ be the all-pass functions associated with symmetric (or (quasi) antimetric) discrete lossless two-port transfer function matrices $\Sigma$, $\Sigma'$ and $\Sigma''$ respectively. Then $\Sigma = \Sigma'\Sigma''$ if and only if $L_1 = L_1'L_1''$ and $L_2 = L_2'L_2''$ hold true.

Proof: Expressing $\Sigma = \Sigma'\Sigma''$ in terms of the corresponding $L_1$ and $L_2$ via (3.7.7) (or (3.7.7')) and its counterparts for $\Sigma'$ and $\Sigma''$, it follows that the factorability condition $\Sigma = \Sigma'\Sigma''$ is equivalent to $L_1 = L_1'L_1''$, $L_2 = L_2'L_2''$.

Since it can be easily shown by pursuing methods outlined in [9] that any

rational all-pass function L can, in fact, be expressed as $L=(\hat{P}/P)\underline{z}^{\underline{m}}$ in irreducible form, where P is a scattering Schur polynomial, Theorem 3.7.1 conveys the important fact that the factorability of multidimensional symmteric or (quasi) antimetric discrete lossless two-port transfer function matrices can be simply expressed in terms of factorability of two scattering Schur polynomials.

## 3.8. SUMMARY:

The present work has been motivated by the possibility of designing structurally passive multidimensional digital filters. A simple algorithm involving the examination of solution of a set of linear simultaneous equations for studying the synthesizability of an arbitrary multidimensional discrete lossless two-port has been derived via factorization of the associated chain matrix $\Theta$, hybrid matrix $\Gamma$ and transfer function $\Sigma$ by introducing a generalized lossless two-port matrix $\Phi$, which in turn can be considered as a multidimensional version of the sigma-lossless transfer functions discussed in the 1-D literature. It turns out that under a generic situation, synthesis not be feasible. In the special case of one-dimension our algorithm provides new methods of realizing structurally passive filters directly in the digital domain. Although in the multidimensional ($k>1$) case synthesis may not be feasible for an arbitrary discrete lossless $\Theta$, $\Gamma$, and $\Sigma$, the possibility of synthesis for special classes of discrete lossless two-ports is by no means ruled out. Examples of such subclasses of two-ports such as the symmetric or the (quasi) antimetric discrete lossless two-ports have been discussed. Existence of other classes of discrete lossless two-ports admitting synthesis, albeit in special topological structures, seems feasible, but remains to be identified. This is especially true in view of cascade synthesizability of certain classes of two-dimensional continuous time systems arising in studies of lumped-distributed netwoks [22]. It may be noted that the cascade synthesizability of lumped-distributed networks can be characterized in terms of properties of certain polynomial matrices having the structure of bigradients (otherwise called resultants). The occurrence of polynomial matrices of similar type has been noted in our study in the context of computing a solution to the fundamental equation (cf. equation (3.7.3)). However, further investigation is needed to explore this connection in successfully utilizing the results of lumped-distributed network theory in multidimensional digital filter design.

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# CHAPTER 4

# SYNTHESIS AND DESIGN OF STRUCTURALLY PASSIVE FULLY RECURSIVE 2-D DIGITAL FILTERS

## 4.1. INTRODUCTION

Various recursive schemes have been proposed in the multidimensional (m-D) digital filter literature. Among these the most widely studied are the quarter plane, the asymmetric and the symmetric half-plane recursive scheme. More recently, motivated by needs for parallel processing of 2-D signals a scheme known as the fully recursive half-plane scheme has been proposed in [15], and a method of designing transfer functions of filters having this recursive structure has been outlined in [5]. The impulse responses of the class of filters just mentioned satisfies the characteristic property that the region of support is a half-plane and the filter is recursive in both horizontal and vertical direction. More specifically, the recursion equation describing the relation between the input x and output y of a filter of this type is given by:

$$L_0^b[\{y_n(m)\}] = -\sum_{i=1}^{L_D} L_i^b[\{y_{n-1}(m)\}] + \sum_{i=0}^{L_N} L_i^a[\{x_{(n-1)}(m)\}] \qquad (4.1.1)$$

where $x_n(m)$, $y_n(m)$ denote the n-th row of the input and the output signal; thus, for example, $x_n(m) = x(m,n)$ for $m = 0, \pm 1, \pm 2, \ldots$etc; and the (row) operations $L_i^a[.]$ and $L_i^b[.]$ respectively denote 1-D linear shift invariant convolution operations with the 1-D sequences $a_i(m)$ and $b_i(m)$. Considering the 2-D Z-transform of (4.1.1), and assuming that the operations $L_i^a[.]$ and $L_i^b[.]$ are all rational we then have (4.1.2) for the transfer function of the filter, where the rational functions $A_i(z_1)$, $B_i(z_1)$ representing the row operations just mentioned are expressed in irreducible rational form (i.e., as the ratio of two relatively prime polynomials) as in (4.1.3).

$$Y(z_1,z_2)/X(z_1,z_2) = \sum_{i=0}^{L_N} A_i(z_1)z_2^i \Big/ \sum_{i=0}^{L_D} B_i(z_1)z_2^i \qquad (4.1.2)$$

$$A_i(z_1) = N_i^a(z_1)/D_i^a(z_1), \; i = 0,1,\ldots L_N \qquad (4.1.3a)$$

$$B_i(z_1) = N_i^b(z_1)/D_i^b(z_1), \; i = 0,1,\ldots L_D \qquad (4.1.3b)$$

On the otherhand it is now well known that an input-output description such as the one expressed in (4.1.2), (4.1.3) is not enough for the successful operation of a digital filter but structural considerations need to be taken into account. The class of structurally passive filters variously known as the wave digital filters [16], orthogonal filters [17] or the lossless bounded real filters [18], when properly designed, are known to satisfy the properties of insensitivity to coefficient perturbation and non-linear arithmatic conditions resulting from overflow, finite precision arithmatic etc. Although much progress has been documented in the synthesis and design of 1-D structurally passive filters, methods for two and higher dimensions are still evolving. Synthesis methods for two and multi-dimensional wave digital filters, which are quarter plane type filters have been reported in [16], [7]. Quarter plane and asymmetric half-plane generalizations of 1-D lattice filters which are, in fact, structurally passive, have been discussed recently in the context of random field modeling in [11],[19].

Following 1-D, in the present paper (pseudo) passive or (pseudo) lossless fully recursive half-plane 2-D digital filters are introduced and a method of their structurally passive synthesis and subsequently that of their design is discussed for the first time. The problem of synthesis of quarter plane causal (thus, including filters causal in a convex cone [20]) structurally passive multidimensional filters of the type mentioned above can be equivalently viewed as the classical network theoretic problem of synthesizing a lossless but otherwise arbitrarily prescribed multidimensional transfer function as an interconnection of elementary building blocks such as capacitors and inductors. This latter problem is completely unresolved in multidimensions ($m>2$), whereas in 2-D synthesis is feasible only in an unconstrained topological structure [20]. On the otherhand, it has been shown that if certain ladder-like constraints are imposed on the structure in which the filter is to be synthesized then the prescribed 2-D transfer function must satisfy further restrictions in addition to input-output losslessness [21], [22], [23]. Related otner synthesis results [7], [16] in this context deal with important special cases when the multidimensional frequency response of the filter possesses certain symmetries. In contrast, the present work provides us with a synthesis of arbitrary lossless fully recursive half-plane 2-D filters.

Additionally, unlike the quarter plane case referred to earlier the synthesis is obtained in a fixed predetermined structure potentially useful for practical implementation.

As in most passive or lossless filter design techniques our synthesis method proceeds by viewing the prescribed passive transfer function as being embedded into the transfer function of a lossless two-port. The synthesis of this fully recursive half-plane lossless two-port takes advantage of a recent algorithm for the design of structurally passive 1-D filters advanced by Rao and Kailath [6] as an extension of the celebrated Schur algorithm [9]. Unlike all other methods known for the synthesis of 1-D continuous as well as discrete lossless two-ports including those available in the classical circuit theoretic literature, the algorithm of [6] enjoys the unique feature that given a transfer function associated with the lossless two-port the synthesis algorithm makes use of <u>rational</u> arithmatic operations only (i.e., nonrational arithmatic operations such as polynomial factorization is not required) [10]. The synthesis method for fully recursive half- plane filters to be presently described fully exploits this rational character of the 1-D algorithm in [6]. Although the details of the method differ nontrivially form 1-D due to considerations characteristic of multidimensional problems (e.g., those utilizing techniques from elementary algebraic curve theory [3], [12]), the synthesis to be outlined can be considered, at least at a conceptual level, to be a generalization of the result in [6] to two-port transfer functions the coefficients of numerator and denominator polynomials of which belong to a field of rational functions (instead of the field of rational numbers). From a different perspective the present work can also be viewed as a generalization of 1-D Schur algorithm to 2-D fully recursive half-plane filters, thus making it possible to cast the present discussion in the closely related framework of modeling of stationary random fields and scattering theory [9].

A note regarding the stability of the filter is in order. The region of analyticity of the transfer function of our filter will be found to marginally differ from those previously considered in the 2-D half-plane literature [4], [5]. This is primarily due to the fact that the results such as those in [4], [5] are motivated by bounded-input-bounded- output considerations, whereas, in contrast, our results are driven by passivity considerations. The fact that

this difference in consideration does indeed lead to diverging formulations of stability in multidimensions ($m>1$), but not in 1-D, is now known [1], [2]. Thus, there is no contradiction between our stability results and those existing in the half-plane literature so far.

In Section 4.2 the fully recursive half-plane passive one-ports are characterized in terms of their transfer function. Similar considerations in the context of two-port transfer functions form the context of Section 4.3. A representation theorem for fully recursive half-plane lossless two-ports analogous to that of the Belevitch canonical form [8] of representation for lossless 1-D continuous two-ports of classical network theory is developed here. In Section 4.4 the synthesis method based on this representation theorem is described, and in Section 4.5 a design methodology is proposed by taking into account the symmetry requirements [14] on the frequency response imposed by many practical multidimensional processing tasks.

## 4.2. FULLY RECURSIVE SYMMETRIC HALF-PLANE PASSIVE SYSTEMS:

By associating the (pseudo) energy $\Sigma\Sigma|x(n_1,n_2)|^2$ to the input $x(n_1, n_2)$ of the system we first develop conditions necessary for the transfer function of a fully recursive half-plane filter to be passive in the sense that:

$$\Sigma\Sigma\ |y(n_1, n_2)|^2 \leq \Sigma\Sigma\ |x(n_1, n_2)|^2 \tag{4.2.1}$$

for any choice of square summable input sequence $x(n_1, n_2)$.

By choosing $x(n_1, n_2) = \delta(n_1, n_2)$ i.e., the 2-D impulse function, the impulse response $h(n_1, n_2)$ of the filter can be obtained as the corresponding output. We then have:

$$y(n_1, n_2) = h(n_1, n_2) = \sum_{k=0}^{\infty} h_k(n_1)\ \delta(n_2-k) \tag{4.2.2}$$

where $h_k(n_1)$, $k=0,1,\ldots$etc. are certain 1-D sequences and $\delta(.)$ is the 1-D impulse sequence.

Considering the z-transform of (4.2.2) we obtain:

$$H(z_1, z_2) = \sum_{k=0}^{\infty} [\sum_{n_1} h_k(n_1) z_1^{n_1}]\ z_2^k \tag{4.2.3}$$

Using the Schwartz inequality it follows from (4.2.3) that

$$|H(z_1, z_2)|^2 \leq k(z_2)\ .\ \sum_{k=0}^{\infty} |\sum_{n_1} h_k(n_1) z_1^{n_1}|^2 \tag{4.2.4}$$

where $k(z_2) = 1 + |z_2|^2 + |z_2|^4 + \ldots$. etc.

If we consider the special case $z_1 = \exp(j\omega_1)$ then we have (4.2.5) from (4.2.4).

$$|H(e^{j\omega_1}, z_2)|^2 \leq k(z_2)\ .\ \sum_{k=0}^{\infty} |H_k(\omega_1)|^2 \tag{4.2.5}$$

where $H_k(\omega_1)$ is the Fourier transform of $h_k(n_1)$ for each $k = 0,1,2,\ldots$etc.

On the otherhand, substituting $z_i = \exp(j\omega_i)$, $i = 1,2$ in (4.2.3) we obtain that the Fourier transform $H(\omega_1,\omega_2)$ of $h(n_1,n_2)$ is given by:

$$H(\omega_1, \omega_2) = \sum_{k=0}^{\infty} H_k(\omega_1)\, e^{jk\omega_2} \tag{4.2.6}$$

which in turn yields that:

$$|H(\omega_1, \omega_2)|^2 = \sum_{k=0}^{\infty} |H_k(\omega_1)|^2 + \text{terms involving } e^{j\omega_2 n} \text{ with } n\neq 0 \tag{4.2.7}$$

Assuming termwise integrability of the right hand side of (4.2.7) we then have:

$$\int_{-\pi}^{+\pi}\int_{-\pi}^{+\pi} |H(\omega_1,\omega_2)|^2 d\omega_1 d\omega_2 = 2\pi \int_{-\pi}^{\pi} \{ \sum_{k=0}^{\infty} H_k(\omega_1)|^2\}\, d\omega_1 \tag{4.2.8}$$

However, by the 2-D Parseval's formula the left hand side of (4.2.8) is equal to $(4\pi^2)\Sigma|h(n_1, n_2)|^2$ i.e., the total (pseudo) energy in the signal $h(n_1, n_2)$, whereas the right hand side can be similarly interpreted as the sum of the (pseudo) energies contained in the row outputs $h_0(n_1)$, $h_1(n_1)$,...etc. Furthermore, if we assume the fully recursive filter under consideration to be passive then from (4.2.1), first part of (4.2.2), and the fact that $x(n_1,n_2) = \delta(n_1,n_2)$, it follows that $\Sigma\Sigma\ |h(n_1,n_2)|^2 < \infty$. Thus, the integral in the left hand side of (4.2.8) is finite, and consequently, except possibly isolated values of $\omega_1$ in the interval $[-\pi,\pi]$, the integrand in the right hand side is bounded i.e., we have:

$$\sum_{k=0}^{\infty} |H_k(\omega_1)|^2 < \infty \tag{4.2.9}$$

In view of (4.2.5) and the fact that $k(z_2) < 1$ for $|z_2| < 1$ we then conclude that for all $|z_1| = 1$ except possibly isolated values on the unit circle and for all $|z_2| < 1$, $H(z_1,z_2)$ is bounded; and furthermore, if $H(e^{j\omega_{10}}, z_{20})$ is unbounded for some $|z_{20}| < 1$ and real $\omega_{10}$, then $\sum_{k=0}^{\infty} |H_k(\omega_{10})|^2$ must be unbounded, and thus $H(e^{j\omega_{10}},z_2)$ must also be so for all $z_2$.

We digress temporarily to examine a consequence of passivity reflected on the

transfer function $H(z_1, z_2)$ of the filter. By using the 2-D Parseval's theorem, (4.2.1) and the fact that $Y(\omega_1,\omega_2) = H(\omega_1,\omega_2)X(\omega_1,\omega_2)$, where $Y(\omega_1,\omega_2)$, $X(\omega_1,\omega_2)$ are the respective Fourier transforms of $y(n_1,n_2)$, $x(n_1,n_2)$ we have that

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |X(\omega_1,\omega_2)|^2 \; (1-|H(\omega_1,\omega_2)|^2) \; d\omega_1 \; d\omega_2 \geq 0 \tag{4.2.10}$$

Since (4.2.10) is true for any input $X(\omega_1,\omega_2)$, we have that $|H(\omega_1,\omega_2)| \leq 1$ for all real two-tuples $(\omega_1,\omega_2)$ except possibly for finitely many of them.

Consequently, if $H(e^{j\omega_{10}},z_{20})$ is unbounded for some $|z_{20}| < 1$ and real $\omega_{10}$, then as shown previously $H(e^{j\omega_{10}},z_2)$ would be unbounded for all $z_2$, and thus for all $z_2$ on $|z_2| = 1$ in particular. This latter situation would then violate the conclusion of the last paragraph. Thus, $H(z_1,z_2)$ is bounded for all $|z_1| = 1$ and $|z_2| < 1$.

We next make the simplifying assumption that for each $k = 0,1,\ldots$etc. $H_k(z_1) = \sum_{n_1} h_k(n_1)z_1^{n_1}$ i.e., the z-transform of $h_k(n_1)$ are rational functions in $z_1$. Since the recursion equation for the fully recursive symmetric half-plane filter is given by (4.1.1) it is clear that the transfer function $H(z_1,z_2)$ is a rational function of $z_2$. Under the present assumption however, $H(z_1,z_2)$, in view of (4.2.3), becomes as in (4.1.2) and (4.1.3) a rational function of both $z_1$ and $z_2$, and can be expressed as the ratio of two relatively prime polynomials $b(z_1,z_2)$ and $a(z_1,z_2)$ as:

$$H(z_1,z_2) = \frac{b(z_1,z_2)}{a(z_1,z_2)} \tag{4.2.11}$$

We now claim that for passive systems presently under consideration, the polynomial $a(z_1,z_2)$ in (4.2.11) cannot have infinitely many zeros on the distinguished boundary $|z_1| = |z_2| = 1$ of the unit bi-disc. For, if $a(z_{10},z_{20}) = 0$ for some $|z_{10}| = |z_{20}| = 1$ then in view of (4.2.11), in order for $|H(\omega_1,\omega_2)| \leq 1$ to be satisfied we would need $b(z_{10},z_{20}) = 0$ i.e., $a(z_1,z_2)$ and $b(z_1,z_2)$ would have a common zero on $|z_1|=|z_2|=1$. However, the presence of

infinitely many such zeros would, in view of Bezout's theorem in algebraic curve theory [3], require that $b(z_1,z_2)$ and $a(z_1,z_2)$ have a common factor, which has been hypothesized to be absent in (4.2.11).

A further consequence of passivity is that when the transfer function $H(z_1,z_2)$, is expressed in terms of ratio of two relatively prime polynomials $b(z_1,z_2)$ and $a(z_1,z_2)$ as in (4.2.11), it must be true that $a(z_1,z_2)$ is relatively prime with the polynomial $\hat{a}(z_1,z_2)$ defined as:

$$\hat{a}(z_1,z_2) = \tilde{a}(z_1,z_2)z_1^{d_1}z_2^{d_2};\ \tilde{a}(z_1,z_2) = a^*(z_1^{*-1},z_2^{*-1}) \qquad (4.2.12\ a,b)$$

where $d_1$, $d_2$ are the partial degrees of a in $z_1$ and $z_2$, and * denotes complex conjugation. To prove this let $g(z_1,z_2)$ be the gcd between $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$. Then as shown in [2] we must have $\hat{g}(z_1,z_2) = \gamma g(z_1,z_2)$ where $\gamma$ = constant, $|\gamma|=1$; and $g(z_1,z_2)$ is either a constant or must have infinitely many zeros on $|z_1| = |z_2| = 1$. In the latter case, $a(z_1,z_2)$ must have infinitely many zeros on $|z_1| = |z_2| = 1$, which has been shown to be impossible in the last paragraph. Thus, $g(z_1,z_2)$ = constant, and $a(z_1,z_2)$ is relatively prime with $\hat{a}(z_1,z_2)$.

The essential features of the above discussion are summerized in the following result.

<u>Property 4.2.1</u>: The transfer function of a passive fully recursive symmetric half-plane filter, when expressed in irreducible rational form as in (4.2.11), satisfies the following two conditions: (i) $a(z_1,z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| < 1$ i.e., $H(z_1,z_2)$ is analytic in $|z_1| = 1$ and $|z_2| < 1$. (ii) $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ as defined in (4.2.12), do not have a common factor.

The condition (ii) in the above property can, in fact, be replaced by any one of the conditions expressed in the following.

<u>Assertion 4.2.1</u>: Let $a(z_1,z_2)$ be a polynomial in $z_1$ and $z_2$ such that $a(z_1,z_2) \neq 0$ for $|z_1| = 1$ and $|z_2| < 1$. Then the following conditions are all equivalent.

(a) $a(z_1,z_2)$ does not have infinitely many zeros on the distinguished boundary $|z_1| = |z_2| = 1$.

(b) $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ are relatively prime polynomials.

(c) Each irreducible factor of $a(z_1,z_2)$ has at least one zero in the domain $|z_1| = 1$, $|z_2| > 1$.

Proof: It has already been proved that (a) implies (b). To show that (b) implies (a) observe that if for some $|z_{10}| = |z_{20}| = 1$, $a(z_{10},z_{20}) = 0$ then from (4.2.12) $\hat{a}(z_{10},z_{20}) = 0$. Consequently, if $a(z_1,z_2)$ has infinitely many zeros on $|z_1| = 1$ then $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ would have infinitely many common zeros (on $|z_1| = |z_2| = 1$). Therefore, due to Bezout's theorem [], $a(z_1,z_2)$ and $\hat{a}(z_1,z_2)$ would not then be relatively prime polynomials. Thus, (a) and (b) are equivalent.

Next, if $a_1(z_1,z_2)$ is any irreducible factor of $a(z_1,z_2)$ then obviously $a_1(z_1,z_2) \neq 0$ or $|z_1| = 1$, $|z_2| < 1$. Furthermore, if $a_1(z_1,z_2)$ does not contain any zero in $|z_1| = 1$, $|z_2| > 1$ then for any $z_1$ on $|z_1| = 1$, $a_1(z_1,z_2) \neq 0$ in $|z_2| < 1$ as well as in $|z_2| > 1$, and thus the values of $z_2$ such that $a_1(z_1,z_2) = 0$ must be on $|z_2| = 1$. Consequently, $a_1(z_1,z_2)$, and thus $a(z_1,z_2)$ would have infinitely many zeros on $|z_1| = |z_2| = 1$. Therefore, (a) (or equivalently (b)) implies (c).

To prove that (c) implies (b) let $g = \gcd(a,\hat{a})$ i.e., $a = g.e$, $\hat{a} = gf$, where e and f are relatively prime polynomials. Then, as shown in [2] $\hat{g} = \gamma g$ where $\gamma$ is a constant. Assuming g to be a nonconstant polynomial, if each irreducible factor of a contains at least one zero in $|z_1| = 1$, $|z_2| > 1$ then g and thus $\hat{g} = \gamma g$ must have a zero in $|z_1| = 1$, $|z_2| > 1$. However, this implies that the polynomial g and thus, in view of $a = ge$, the polynomial a must have a zero in $|z_1| = 1$, $|z_2| < 1$, which is a contradiction. Thus, g = constant and a and $\hat{a}$ are relatively prime.

We also have the following important result.

Property 4.2.2a: If a rational function $H=H(z_1,z_2)$ as in (4.2.11) is such that

$|H| \leq 1$ on $|z_1| = |z_2| = 1$ except possibly at finite number of points where H is not well defined and if H satisfies the conditions expressed in property 4.2.1(i) (thus, if $H(z_1, z_2)$ is tranfer function of a passive fully recursive half plane filter) then $|H| \leq 1$ for all $|z_1| = 1$ and $|z_2| < 1$. Furthermore, for some $(z_{10}, z_{20})$ with $|z_{10}| = 1, |z_{20}| < 1$. Then $H(z_{10}, z_2)$ is a constant independent of $z_2$. Assuming H to be nonconstant, the latter situation can arise for at most finitely many values of $z_{10}$ (with $|z_{10}| = 1$).

<u>Proof</u>: Due to property 4.2.1 (i) the denominator polynomial of $H(z_1, z_2)$ cannot be zero for some fixed $|z_{10}| = 1$ and arbitrary values of $z_2$. Thus, if for any $z_{10}$ with $|z_{10}| = 1$ we define $H_1 = H_1(z_2) = H(z_{10}, z_2)$ then due to our hypothesis $H_1$ is well defined, analytic in $|z_2| < 1$ and $|H_1| \leq 1$ for all $|z_2| = 1$ (except at the possible poles). Thus, by maximum modulus theorem $|H_1| \leq 1$ for all $|z_2| < 1$. Since this is true for arbitrary $z_{10}$ on $|z_1| = 1$ the first part follows.

To show the second part assume for contradiction that for some $|z_{10}| = 1$, $|z_{20}| < 1$, we have $H(z_{10}, z_{20}) = 1$. Then as shown above the maximum modulus theorem applies to $H_1 = H_1(z_2) = H_1(z_{10}, z_2)$ and thus $|H_1(z_{20})| = 1$ with $|z_{20}| = 1$ implies that $H_1 = H_1(z_2) = C = \text{constant}$ i.e., in view of (4.2.1) $b(z_{10}, z_2) = C\ a(z_{10}, z_2)$. However, if H is nonconstant then $b(z_1, z_2)$ and $a(z_1, z_2)$ are relatively prime nonconstant polynomials. Consequently, $b(z_{10}, z_2)$ and $a(z_{10}, z_2)$ treated as one-variable polynomials may fail to be relatively prime for at most finitely many values of $z_{10}$ [1]. The second part is thus established.

In fact, the following result in Property 4.2.2b can also be proved. This result shows that the polynomials of the type described in properties 4.2.1 (i) and 4.2.1 (ii) characterize denomination of irreducible rational functions satisfying the (half-plane) boundedness property: $|H| \leq 1$ for $|z_1| = 1$, $|z_2| < 1$.

<u>Property 4.2.2b</u>: If H is a nonconstant irreducible rational function as expressed in (4.2.11) and is such that $|H| \leq 1$ for $|z_1| = 1, |z_2| < |$ then either a is a constant or satisfies properties 4.2.1 (i) and 4.2.1 (ii).

<u>Proof:</u> Obviously it is impossible to have $a = 0$ and $b \neq 0$ for any $|z_1| = 1, |z_2| < 1$, because otherwise $|H|$ would be unbounded there. If $a = b = 0$ for some $|z_{10}| = 1$, $|z_{20}| < 1$ then consider an arbitrary small arc $\Gamma_1$ of $|z_1| = 1$ issuing from $z_{10}$.

Let $\Gamma_2$ be the continuous [12] arc traced out by $z_2$ (beginning from $z_{20}$) such that $a(z_1,z_2)=0$ is satisfied. Note that since $\Gamma_1$ is assumed arbitrarily small, due to the continuity property of zeros of a polynomial as a function of its coefficients, $\Gamma_2$ must lie completely within $|z_2|<1$. We next claim that for $z_1 \varepsilon \Gamma_1$ there must exist values of $z_2 \varepsilon \Gamma_2$ such that $a=0$, $b\neq 0$, because otherwise a and b would have infinitely many common zeros, which due to Bezout's theorem [3] violate the fact that a and b are relatively prime polynomials. However, since $\Gamma_1 \subset \{z_1, |z_1|=1\}$ and $\Gamma_2 \subset \{z_2; |z_2|<1\}$ this latter conclusion has already been shown to be impossible. Thus $a\neq 0$ for $|z_1|=1, |z_2|<1$.

Finally, if $a(z_{10},z_{20})=0$ for some $|z_{10}|=|z_{20}|=1$ then $b(z_{10},z_{20})=0$ because otherwise $|H|\leq 1$ would be violated in $|z_1|=1, |z_2|<1$ at the vicinity of $(z_{10},z_{20})$. Thus existence of infinitely many such $(z_{10},z_{20})$ would again violate the relative primeness of a and b.

We next assume the filter to be (pseudo) <u>lossless</u> in the sense described earlier i.e., equations (4.2.1) and (4.2.10) are satisfied with equality. Consequently, from (4.2.10) we then have that for all 2-tuples $(\omega_1,\omega_2)$ with the possible exception of finitely many (4.2.13) holds true.

$$|H(\omega_1,\omega_2)| = 1 \qquad (4.2.13)$$

We first note that the rational transfer function $H(z_1,z_2)$ of a (pseudo) lossless fully recursive half-plane transfer function satisfies the property that

$$H(z_1,z_2)\,\tilde{H}(z_1,z_2) = 1 \qquad (4.2.14)$$

To substantiate this result we observe from the definition of the operation $\tilde{}$ that $\tilde{H}(z_1,z_2) = H^*(z_1,z_2)$ for $|z_1|=|z_2|=1$. Consequently, from (4.2.13) it follows that $\tilde{H}(z_1,z_2)=H^{-1}(z_1,z_2)$ for all 2-tuples $(z_1,z_2)$ on $|z_1|=|z_2|=1$ with possible exception of at most finitely many values. Thus the two variable rational function $\tilde{H}(z_1,z_2)$ and $H^{-1}(z_1,z_2)$ assume equal values at infinitely many distinct points $(z_1,z_2)$, and consequently, due to analytic continuation are identically same, i.e., $\tilde{H}(z_1,z_2) = H^{-1}(z_1,z_2)$ for all $z_1$ and $z_2$.

For convenience of further exposition the following terminology will be introduced. Any rational function $H(z_1,z_2)$ as expressed in (4.2.11) will be said to be a <u>fully recursive half-plane all-pass function</u> if $H(z_1,z_2)$ satisfies the conditions stated in property 4.2.1 and in equation (4.2.14). Thus, transfer functions of (pseudo) lossless fully recursive half-plane filters are fully recursive half-plane all-pass functions.

A function $A(z_1,z_2)$ of two variables $z_1$, $z_2$, when expressible as a polynomial in $z_2$ with coefficients as rational functions in $z_1$ will be said to be a <u>pseudopolynomial</u> (in $z_2$). Thus, if $A(z_1,z_2)$ is a pseudopolynomial then

$$A(z_1,z_2) = \alpha_0(z_1) + \alpha_1(z_1)z_2 + \ldots + \alpha_{N_2}(z_1)z_2^{N_2} \qquad (4.2.15)$$

where $\alpha_k(z_1)$'s are rational functions in $z_1$. With $A(z_1,z_2)$ as given in (4.2.15), where $\alpha_{N_2}(z_1)$ is not identically zero, the integer $N_2$ will also be denoted by $\deg_2 A$. Furthermore, the notation $\bar{A}(z_1,z_2)$ will be used to denote the pseudo- polynomial obtained from $A(z_1,z_2)$ as:

$$\bar{A}(z_1,z_2) = \tilde{A}(z_1,z_2)z_2^{N_2} \qquad (4.2.16)$$

Two pseudopolynomials $B(z_1,z_2)$ and $C(z_1,z_2)$ are said to be coprime if there is no pseudopolynomial $D(z_1,z_2)$ actually involving $z_2$ such that $B(z_1,z_2) = D(z_1,z_2)\, B_1(z_1,z_2)$ and $C(z_1,z_2) = D(z_1,z_2)\, C_1(z_1,z_2)$ for some pseudopolynomials $B_1(z_1,z_2)$ and $C_1(z_1,z_2)$. The following property then holds true.

<u>Property 4.2.3</u>: Any fully recursive half-plane all-pass function $H(z_1,z_2)$ (thus, rational transfer function of (pseudo) lossless fully recursive half-plane filter) can be expressed as follows:

$$H(z_1,z_2) = -D(z_1)\,[A(z_1,z_2)/\bar{A}(z_1,z_2)] \qquad (4.2.17)$$

where i) $A(z_1,z_2)$ is a pseudopolynomial

ii) $D(z_1) = z_1^N \gamma\, [d(z_1)/\hat{d}(z_1)]$, where $d(z_1)$ is a polynomial in $z_1$, $\gamma$ is a constant of unit modulus and N = integer.

iii) the pseudopolynomials $A(z_1,z_2)$ and $\tilde{A}(z_1,z_2)$ are coprime,

iv) $\tilde{A}(z_1,z_2) \neq 0$ for all $|z_1|=1$, $|z_2|<1$.

Conversely, any rational function expressible as in (4.2.17) with (i), (ii), (iii) and (iv) in force is a fully recursive half-plane all-pass function.

Proof of property 4.2.3: Let $H(z_1,z_2) = A(z_1,z_2)/B(z_1,z_2)$, where $A = A(z_1,z_2)$ and $B = B(z_1,z_2)$ are pseudopolynomials expressible as $A = a_N/a_D$ and $B = b_N/b_D$, where in turn $a_N = a_N(z_1,z_2)$, $b_N = b_N(z_1,z_2)$ are polynomials in both $z_1$ and $z_2$, whereas $a_D = a_D(z_1)$ and $b_D = b_D(z_1)$ are polynomials in $z_1$ only.

We further assume that $H = H(z_1,z_2)$ expressed as in (4.2.18) is in irreducible rational form i.e., the pairs of polynomials $(a_N,a_D)$, $(b_N,b_D)$, $(a_N,b_N)$ and $(b_D,a_D)$ are relatively prime.

$$H = (b_D a_N)/(a_D b_N) \tag{4.2.18}$$

Then from (4.2.18) equations (4.2.19), (4.2.20) and (4.2.21) follows, where the generic notation $n_{ip}$ for denoting the degree of the polynomial p in the i-th variable has been used.

$$H = \tilde{H}^{-1} = (\tilde{a}_D\tilde{b}_N)/(\tilde{b}_D\tilde{a}_N) = [(\hat{a}_D\hat{b}_N)/(\hat{b}_D\hat{a}_N)]z_1^{\nu_N-\nu_D} z_2^{N_1} \tag{4.2.19}$$

$$\nu_N = n_{1b_D} + n_{1a_N} \geq 0, \quad \nu_D = n_{1a_D} + n_{1b_N} \geq 0 \tag{4.2.20 a,b}$$

and $N_1 = n_{2a_N} - n_{2b_N}$ (4.2.21)

Since $H$ is analytic in $|z_1|=1$, $|z_2|<1$ and neither $\hat{a}_D$ nor $\hat{b}_N$ can have a factor $z_2$, it clearly follows that $N_1 \geq 0$. Also, since $H$ in (4.2.18) is in irreducible rational form, it follows by comparing (4.2.18) and 2.19) that

$$\alpha b_D a_N = \hat{a}_D\hat{b}_N z_1^{\nu_N} z_2^{N_1}, \quad \alpha a_D b_N = \hat{b}_D\hat{a}_N z_1^{\nu_D} \tag{4.2.22 a,b}$$

where $\alpha = \alpha(z_1,z_2)$ is a polynomial in $z_1$ and $z_2$. By inserting (4.2.22b) into (4.2.18) and subsequently making use of the relations between $\hat{a}_N$ and $\tilde{a}_N$,

between $\hat{a}_D$ and $\tilde{a}_D$ and finally by using $A = a_N/a_D$, (4.2.20) and (4.2.21) we obtain the following

$$H = \alpha[(a_D b_D)/(\hat{a}_D \hat{b}_D)][A/\tilde{A}].z_1^{-(n_{1a_N} + n_{1b_N})} z_2^{n_{2a_N}} \tag{4.2.23}$$

By defining $d = a_D b_D$ and noting the fact that $\bar{A} = \tilde{A}\, z_2^{n_{2b_N}}$ we then have:

$$H = \alpha(d/\hat{d})(A/\bar{A})\, z_1^{-(n_{1a_N} + n_{1b_N})} \tag{4.2.24}$$

Since $H$ in (4.2.18) is irreducible and analytic in $|z_1|=1, |z_2|<1$ we note that $a_D b_N$ cannot have a factor $z_2$. Invoking this fact and considering the $\hat{}$ of (4.2.22a) we then have:

$$\hat{\alpha}\hat{b}_D\hat{a}_D = a_D b_N z_1^{-k} \tag{4.2.25}$$

where in (4.2.25) k is the total multiplicity of $z_1$ in $(a_D b_N)$. By substituting (4.2.25) into (4.2.22b) we obtain $\alpha\hat{\alpha} = z_1^{\nu_D - k}$ (note that since from (4.2.20b) $\nu_D$ = degree of $(a_D b_N)$ in $z_1$ it obviously follows that $\nu_D - k \geq 0$). Consequently, it must be true that $\alpha$ is a monomial involving $z_1$ only i.e., is of the form

$\alpha = \gamma z_1^{\nu_D - k}$ for some constant $\gamma$. Then $\alpha\hat{\alpha} = \gamma\gamma^* = z_1^{\nu_D - k}$. Thus, $|\gamma|=1$. Therefore, (4.2.24) yields (4.2.17) with $N = \nu_D - (n_{1a_N} + n_{1b_N} + k)$. Properties 4.2(i) and 4.2(ii) are thus established. To show that 2(iii) holds true note that

$$\bar{A} = (\hat{a}_N/\hat{a}_D)\, z_1^{n_{1b_D} - n_{1b_N}} \tag{4.2.26}$$

Consequently, if $A$ and $\bar{A}$ has a pseudopolynomial common factor then it follows from $A_N = a_N/a_D$ and (4.2.26) that $a_N$ and $\hat{a}_N$ must have a common factor involving $z_2$. In view of (4.2.22a,b) then $a_D b_N$ and $b_D a_N$ would not be relatively prime, thus violating the irreducibility of H in (4.2.18). Finally, to prove (iv) note that it follows from property 4.2.1, (4.2.18) and (4.2.22b) that $a_D b_N$ and thus $\hat{a}_N$ is nonzero for $|z_1| = 1$, and
$|z_2| < 1$.

The converse proposition follows trivially from the fact that any $H = H(z_1, z_2)$

satisfying (4.2.17) along with (i) through (iv) is necessarily analytic in $|z_1| = 1$, $|z_2| < 1$ and has the property of $H\tilde{H} = 1$ on $|z_1| = |z_2| = 1$.

## 4.3. FULLY RECURSIVE SYMMETRIC HALF-PLANE LOSSLESS TWO-PORTS:

Characterizations of fully recursive symmetric half- plane passive as well as lossless one-port filters have been established in the previous section in terms of the transfer function of the filter. In this section we make use of the results of the previous section to characterize fully recursive symmetric half-plane lossless two-ports. In particular, a convenient representation for such two-ports analogous to the Belevitch canonical representation of continuous time 1-D lossless circuits of classical network theory [8] is developed. This representation is then subsequently used in section 4.4 to synthesize the filter in a specific structure.

A two-port system of the type under consideration is lossless if (4.3.1) holds true for any finite (pseudo) energy inputs $x_1(n_1,n_2)$ and $x_2(n_1,n_2)$.

$$\Sigma\Sigma|y_1(n_1,n_2)|^2 + \Sigma\Sigma|y_2(n_1,n_2)|^2 = \Sigma\Sigma|x_1(n_1,n_2)|^2 + \Sigma\Sigma|x_2(n_1,n_2)|^2 \tag{4.3.1}$$

Considering $x_2(n_1,n_2) = 0$ for all $n_1$, $n_2$ we then have from (4.3.1) that for any finite (pseudo) energy $x_1(n_1,n_2)$:

$$\Sigma\Sigma|y_1(n_1,n_2)|^2 \leq \Sigma\Sigma|x_1(n_1,n_2)|^2, \quad \Sigma\Sigma|y_2(n_1,n_2)|^2 \leq \Sigma\Sigma|x_1(n_1,n_2)|^2 \tag{4.3.2a,b}$$

On the other hand, if $S = S(z_1,z_2) = [S_{ij}(z_1,z_2)]$ is the transfer function of the two-port then for $X_2(z_1,z_2) \equiv 0$ we have $Y_1(z_1,z_2) = S_{11}(z_1,z_2)X_1(z_1,z_2$ and $Y_2(z_1,z_2) = S_{21}(z_1,z_2)X_1(z_1,z_2)$, where as in last section capital letters are used to denote 2-D z-transforms. Thus, due to (4.3.2a) and (4.3.2b) the transfer functions $S_{11} = S_{11}(z_1,z_2)$ and $S_{21} = S_{21}(z_1,z_2)$ are (pseudo) passive, and thus satisfy property 4.2.1. Similary, by considering $x_1(n_1,n_1) = 0$ for all $n_1$, $n_2$ it can be shown that $S_{12} = S_{12}(z_1,z_2)$ and $S_{22} = S_{22}(z_1,z_2)$ satisfy Property 4.2.1.

Furthermore, by considering 2-D Parseval's theorem (4.3.1) can be made to yield

(4.3.3) where the column vector $X(\omega_1,\omega_2) = (X_1(\omega_1,\omega_2)\ X_2(\omega_1,\omega_2))^t$, and * denotes the combined operation of complex conjugation and matrix transposition.

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X^*(\omega_1,\omega_2)(I_2 - S^*(\omega_1,\omega_2)S(\omega_1,\omega_2))X(\omega_1,\omega_2) = 0 \qquad (4.3.3)$$

Since (4.3.3) holds for any $X(\omega_1,\omega_2)$ it follows that for any 2-tuple $(\omega_1,\omega_2)$ except possibly finitely many, we have $S^*(\omega_1,\omega_2)S(\omega_1,\omega_2) = I_2$. This latter equation, by exploiting arguments similar to those used in the paragraph following (4.2.14) yields that for all $z_1$, $z_2$:

$$\tilde{S}(z_1,z_2)\ S(z_1,z_2) = I_2 \qquad (4.3.4)$$

A (2 x 2) rational matrix $S = S(z_1,z_2)$ is said to be <u>fully recursive half-plane lossless bounded</u> if: (i) each entry of S satisfies the conditions expressed in properties 4.2.1 (i), (ii) and equation (4.3.4) holds true.

Note that the transfer function of a fully recursive half- plane lossless two-port is necessarily of the above type. As a consequence of property 4.2.2, we then have the following important conclusion.

<u>Proposition 4.3.1</u>: Each entry of a fully recursive half-plane bounded matrix S $= [S_{ij}]$ satisfies $|S_{ij}| < 1$ for all $|z_1|=1$ and $|z_2|<1$.

<u>Proof</u>: Since on $|z_1| = |z_2|=1$, we have $S^*S = I_2$, and thus $|S_{11}|^2 + |S_{21}|^2 = 1$, $|S_{22}|^2 + |S_{12}|^2 = 1$, which in turn imply $|S_{ij}| < 1$ for all i,j. The result then follows from property 4.2.2.

Consider next a fully recursive half-plane lossless bounded matrix S. Since each entry of S satisfies property 4.2.1, the rational fnction det H also satisfies property 4.2.1. Also, it follows from (4.3.4) that (det S)(det $\tilde{}$) = 1. Thus, S is a fully recursive symmetic half-plane all-pass function as defined in section 4.2 and admits of the representation (4.2.17) described in property 4.2.3, i.e., (4.3.5) holds.

$$\det S = -D.(A/\tilde{A}) \qquad (4.3.5)$$

Consequently, it follows from (4.3.4) and (4.3.5) that

$$\tilde{S}_{11} = -(1/D)(\bar{A}/A)S_{22};\ \tilde{S}_{21} = (1/D)(\bar{A}/A)S_{12} \qquad (4.3.6a,b)$$

$$\tilde{S}_{12} = (1/D)(\bar{A}/A)S_{21};\ \tilde{S}_{22} = -(1/D)(\bar{A}/A)S_{11} \qquad (4.3.7a,b)$$

It may be shown via the operation ~ that (4.3.6a) is identical with (4.3.7b), where as (4.3.6b) is identical with (4.3.7a). Now since $S_{11}$ in (4.3.7b) satisfies property 4.2.1, it follows that when $S_{11}$ is expressed in irreducible rational form, each of the irreducible factors of the denominator polynomial must have zeros in $|z_1|=1$, $|z_2|\tilde{\geq}1$. However, due to property 4.2.1, $S_{22}$ is analytic in $|z_1|=1$, $|z_2|<1$ and thus $\tilde{S}_{22}$ is analytic in $|z_1|=1$, $|z_2|>1$. This is possible, however, only if the denominator polynomial of $S_{11}$ in irreducible rational form is completely cancelled by the numerator of $(\bar{A}/A)$. Thus, (4.3.8a) follows, where B is a pseudopolynomial. By inserting (4.3.8a) in (4.3.7b) and performing the operation ~, (4.3.8b) follows via the use of the identity $D\tilde{D} = 1$. Equations (4.3.9a,b) follow from (4.3.7a) in a similar manner, where again C is a pseudo-polynomial.

$$S_{11} = B/\bar{A},\ S_{22} = -D(\tilde{B}/\tilde{\bar{A}}) \qquad (4.3.8a,b)$$

$$S_{21} = C/\bar{A},\ S_{12} = D(\tilde{C}/\tilde{\bar{A}}) \qquad (4.3.9a,b)$$

Inserting (4.3.8) and (4.3.9) in the expression for (det S) in (4.3.5) we then have

$$A\tilde{A} = B\tilde{B} + C\tilde{C} \qquad (4.3.10)$$

Also, since $S_{22}$ and $S_{12}$ are analytic in $|z_1|=1$, $|z_2|<1$ we have from (4.3.8b) and (4.3.9b) that:

$$\deg_2 B \leq \deg_2 A\ ;\ \deg_2 C \leq \deg_2 A \qquad (4.3.11)$$

The above discussion can be succinctly expressed in the following representation of a fully recursive symmetric half- plane lossless bounded matrix.

<u>Property 4.3.1</u>: Any fully recursive symmetric half-plane lossless bounded

matrix can be represented in terms of three pseudopolynomials A, B and C as in (4.3.8) and (4.3.9), where $\tilde{A}$ is nonzero in $|z_1|=1$, $|z_2|<1$; the pseudopolynomials A and $\tilde{A}$ are coprime; and furthermore (4.3.10), (4.3.11) hold true. Conversely, any matrix, which admits of the above representation is fully recursive symmetric half-plane bounded.

Since property 4.2.1 is automatically satisfied by each entry of a matrix S expressed via (4.3.8), (4.3.9) and (4.3.10), to establish the above converse proposition we only need to observe that straightforward algebraic manipulation with (4.3.8) through (4.3.10) yield (4.3.4). For convenience of exposition any S expressed as in (4.3.8) and (4.3.9) will be referred to as in <u>standard form</u>.

A fully recursive half-plane lossless two-port can be alternatively described by means of a chain matrix $T = T(z_1,z_2)$ defined as in (4.3.12)., where $X_1$, $X_2$ and $Y_1$, $Y_2$ are respective inputs and output signals from ports 1 and 2.

$$\begin{bmatrix} Y_1 \\ X_1 \end{bmatrix} = T \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} \tag{4.3.12}$$

It can be easily shown from (4.3.8) through (4.3.12) that the following property characterizes the chain matrices of the type described above.

<u>Property 4.3.1'</u>: The chain matrix $T = [T_{ij}]$ associated with a fully recursive half-plane two-port is lossless if and only if it can be expressed as

$$T_{11} = DA/C \; ; \; T_{12} = B/C \tag{4.3.13a,b}$$

$$T_{21} = \tilde{D}Bz_2^{\,n}A/C \; ; \; T_{22} = \tilde{A}/C \tag{4.3.14a,b}$$

where $n_A = \deg_2 A$, and A, B, C and D satisfies the same restrictions described in property 4.3.1. Also, any T as in (4.3.13), (4.3.14) is said to be in <u>standard form</u>.

## 4.4. SYNTHESIS OF FULLY RECURSIVE HALF-PLANE LOSSLESS TWO-PORTS:

A procedure for synthesizing fully recursive half-plane lossless two-ports as an interconnection of more elementary building blocks of the same type will be developed in this section. The synthesis algorithm can be viewed as a generalization of the algorithm for synthesizing 1-D discrete lossless two-ports as described by Rao and Kailath in [6]. Our synthesis procedure exploits the unique feature of the algorithm described in [6] that (in 1-D) given (polynomials) A, B, C the arithmetic operations needed to be performed on the coefficients of A, B and C in each cycle of the algorithm requires rational operations only. To the best of our knowledge this is the only algorithm of the above mentioned type available for synthesis of 1-D discrete as well as continuous domain lossless two-ports including those in classical network theory [8]. (All other algorithms known prior to [6] required nonrational operations e.g., polynomial factorization). The basic structure of the filter to be presently synthesized would thus be the same as in [6], whereas the elementary building blocks are certain 1-D two port sections to be referred to as the generalized Gray-Markel sections (GGM section) and $z_2$-type delays, each of which are fully recursive half-plane lossless.

A generalized Gray-Markel section is a 1-D two port as shown in figure 4.4.1 where the 1-D transfer functions (assumed rational) $k_1$ and $k_2$ satisfy the relations:

$$k_1\tilde{k}_1 + k_2\tilde{k}_2 = 1 \tag{4.4.1}$$

and are such that $k_1$ (and thus $k_2$ in view of (4.4.1)) satisfies $k_1\tilde{k}_1 = |k_1|^2 <$ 1 everywhere on $|z_1| = 1$ with the possible exception of finite number of values of $z_1$ for which we may have $k_1\tilde{k}_1 = |k_1|^2 = 1$.

We first note that given any rational function $k_1$ of $z_1$ satisfying the above conditions it is always possible to find a rational function $k_2$ satisfying the same conditions as that of $k_1$ along with (4.4.1). (The role of $k_1$ and $k_2$ can obviously be interchanged in the present considerations). To show this let $k_1 = n_1/d_1$ where $n_1$, $d_1$ are polynomials in $z_1$. Then $(1 - k_1\tilde{k}_1) = N_1/(d_1\tilde{d}_1)$, where

$N_1 = d_1\tilde{d}_1 - n_1\tilde{n}_1$. Thus, $N_1 = \tilde{N}_1$, and for all $z_1$ on $|z_1| = 1$, $N_1(z_1)$ is real and we have that $N_1(z_1) \geq 0$ as a consequence of $k_1\tilde{k}_1 \leq 1$. Therefore, the (spectral) factorization $N_1 = n_2\tilde{n}_2$, where $n_2$ is a polynomial in $z_1$ holds. Also, by (possibly) rearranging the irreducible factors of $(d_1\tilde{d}_1)$ to write $d_1\tilde{d}_1 = d_2\tilde{d}_2$, where $d_2$ = polynomial, we can have $k_2 = n_2/d_2$ such that (4.4.1) is satisfied. Note that since the factorizations $N_2 = n_2\tilde{n}_2$ and $d_1\tilde{d}_1 = d_2\tilde{d}_2$ are not unique the $k_2$ so obtained is not unique unless further restrictions are imposed.

The transfer function matrix $S_G = S_G(z_1)$ associated with such a GGM section can be expressed as in (4.4.2a), whereas the corresponding chain matrix T is given in (4.4.2b).

$$S_G = \begin{bmatrix} k_1 & k_2 \\ \tilde{k}_2 & -\tilde{k}_1 \end{bmatrix} \qquad T_G = (1/\tilde{k}_2)\begin{bmatrix} 1 & k_1 \\ \tilde{k}_1 & 1 \end{bmatrix} \tag{4.4.2a,b}$$

Since $S_G$ in (4.4.2a) satisfies the representation described in property 4.3.1 with $A = 1$, $B = k_1$, $C = \tilde{k}_2$ and $D = 1$ the GGM section is indeed a fully recursive half-plane lossless two-port.

To proceed with the synthesis of a prescribed fully recursive half-plane lossless bounded matrix S or, equivalently, corresponding chain matrix T as described respectively in property 4.3.1 or 4.3.1', we first note that in view of porposition 3.1 the rational function: $k_1 = k_1(z_1) = S_{11}(z_1,0)$ satisfies $|k_1| < 1$ for all $|z_1| = 1$ with the possible exception of finite number of values of $z_1$ where $|k_1|=1$. Therefore, in view of the preceeding discussion $k_1$ defines a GGM section i.e., a $k_2$ can be found such that $|k_2|<1$ everywhere on $|z_1|=1$ with the possible exception of finite number of values of $z_1$, for which $|k_2|=1$ and that (4.4.1) is satisfied.

Step 1: The first step is to extract a GGM section with $k_1 = S_{11}(z_1,0)$ from prescribed S or T as shown in figure 4.4.2. Since a cascade connection of two two-ports amounts to multiplication of the corresponding chain matrices, the chain matrix of the remaining two-port is then $T' = T_G^{-1}T$. From (4.3.13),

(4.3.14) and (4.4.2b) we can write:

$$T' = (1/Ck_2)\begin{bmatrix} D(A-k_1\tilde{B}z_2^{\,n_A}) & B-k_1\bar{A} \\ & \\ D(\tilde{B}z_2^{\,n_A}-\tilde{k}_1A) & \bar{A}-\tilde{k}_1B \end{bmatrix} \qquad (4.4.3)$$

We next define the pseudopolynomials $A'$, $B'$, $C'$ and the 1-D rational function $D'$ as in (4.4.4) and (4.4.5) below, where $p=p(z_1)$ is the conjugate reciprocal[1] polynomial factor of largest degree present in the numerator of $\bar{A}-\tilde{k}_1B$, when expressed in irreducible rational form.

$$\tilde{p}A' = A(1-k_1\tilde{S}_{11}) = A-k_1\tilde{B}z_2^{\,n_A} \;;\; pC' = Ck_2 \qquad (4.4.4a,b)$$

$$pB' = \bar{A}(S_{11}-k_1) = B-k_1\bar{A} \;;\; D' = D(\tilde{p}/p) \qquad (4.4.5a,b)$$

We claim that $deg_2A' = deg_2A$. To prove this, clearly $deg_2A' \le deg_2A$ and note that (4.4.4a) yields $\tilde{p}A'/\tilde{A} = 1-\tilde{k}_1S_{11}$, which implies that if $deg_2A' < deg_2A$ then for arbitrary $z_1$ we would have $\tilde{k}_1(z_1)S_{11}(z_1,0) = |S_{11}(z_1,0)|^2 = 1$. The impossibility of this latter situation has already been demonstrated. As a consequence of this we can write $T'$ as in (4.4.6) and (4.4.7), where $n_{A'} = deg_2A' = deg_2A$.

$$T'_{11} = D'A'/C' \;;\; T'_{12} = B'/C' \qquad (4.4.6a,b)$$

$$T'_{21} = D'\tilde{B}'z_2^{\,n_{A'}}/C' \;;\; T'_{22} = \bar{A}'/C' \qquad (4.4.7a,b)$$

We next claim that the pseudopolynomial $\bar{A}'$ satisfies the properties that $\bar{A}' \ne 0$ for $|z_1|=1$, $|z_2|<1$ and that $\bar{A}'$ is coprime with $A'$. To prove this we write $A' = A'_N/A'_D$ in irreducible rational form, and thus $\bar{A}' = (\hat{A}'_N/\hat{A}'_D).z^{\alpha}$, where $\alpha$ = integer and $\hat{A}'_N/\hat{A}'_D$ is in irreducible rational form. Thus, from the

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[1]A polynomial $p$ is said to be conjugate reciprocal if $\hat{p}=\gamma.p$ for some constant $\gamma$.

definition of $\bar{A}'$ it follows that $\hat{A}'_N$ is devoid of conjugate reciprocal polynomial factors in $z_1$ only. If we assume for the prupose of a proof by contradiction that for some value of $z_1=z_{10}, z_2=z_{20}$ with $|z_{10}|=1$, $|z_{20}|<1$ we have $\bar{A}'=0$ i.e., $\hat{A}'_N=0$ then since $\hat{A}'_N$ cannot have a factor $(z_1-z_{10})$, by changing the value of $z_1$ from $z_{10}$ along an arbitrarily small arc $\Gamma_1$ of the unit circle $|z_1|=1$ it would be possible to find a continuous [12] set $(z_1,z_2)$ of zeros of $\hat{A}'_N$ i.e., also of $\bar{A}'$ with $z_1 \varepsilon \Gamma_1 \subset \{z_1; |z_1|=1\}$ and $|z_2|<1$.

Also, since it follows from (4.4.4a) and $\deg_2 B \leq n_A = n_A'$ that $p\bar{A}'=\bar{A}(1-\tilde{k}_1 S_{11})$ and $\bar{A}\neq 0$ in $|z_1|=1$, $|z_2|<1$ (cf. property 4.3.1) we would then have that for all $z_1 \varepsilon \Gamma_1$ there exists some $z_2$ in $|z_2|<1$. such that $\tilde{k}_1 S_{11}=1$. Since $|k_1|\leq 1, |S_{11}|\leq 1$ for all $|z_1|=1, |z_2|<1$ (cf. Proposition 4.3.1) the last conclusion implies that there exists $z_2$ in $|z_2|<1$ such that $|\tilde{k}_1|=|S_{11}(z_1,0)|=1$ and $|S_{11}(z_1,z_2)|=1$ for all $z_1 \varepsilon \Gamma_1$. However, this last conclusion leads to a contradiction in view of Property 4.2.2a. Thus, $\bar{A}'\neq 0$ i.e., $\hat{A}'_N\neq 0$ for $|z_1|=1, |z_2|<1$.

Also, since due to Property 4.2.2a $|\tilde{k}_1|=|S_{11}(z_1,0)|=1$ is possible for at most finite number of values of $z_1$, for $|z_1|=|z_2|=1$ we have $|S_{11}|\leq 1$ and $\bar{A}\neq 0$ for at most finite number of exceptions on the distinguished boundary of the unit bi-disc, we conclude from $p\bar{A}'=\bar{A}(1-\tilde{k}_1 S_{11})$ that $\bar{A}'$, thus $\hat{A}'_N$ may have at most finite number of zeros on $|z_1|=|z_2|=1$. Since as shown earlier $\bar{A}'_N\neq 0$ in $|z_1|=1, |z_2|<1$ it follows from Assertion 4.2.1 that $A_N$ and $\hat{A}_N$ are relatively prime polynomials. Consequently, the pseudopolynomials $\bar{A}$ and $\bar{A}'$ are relatively prime.

Finally, straightforward algebraic manipulation along with (4.4.1) yields $\tilde{A}\tilde{A}' = \tilde{B}\tilde{B}' + \tilde{C}\tilde{C}'$, whereas $\deg_2 A' \leq \deg_2 B'$, $\deg_2 A' \leq \deg_2 C'$ follow from (4.4.4a,b), (4.4.5), (4.3.11) and $n_A = n_{A'}$. Since, clearly $D'$ as in (4.4.5b) possesses the requisite properties for $T'$ to be in standard form, in view of Property 4.3.1′ all the conditions necessary for $T = [T_{ij}]$, as given in (4.4.6), (4.4.7), to be a fully recursive half-plane lossless two-port chain matrix are satisfied.

We further note that as a consequence of the choice $k_1=S_{11}(z_1,0)$ we have from (4.4.5a) that $B'(z_1,0) = 0$ for arbitrary $z_1$ i.e., the pseudopolynomial $B'$ contains $z_2$ as a factor. Also, if $C$ contains a pseudopolynomial factor $z_2$ then

so does $C'$.

Step 2: In the next step we form a fully recursive half-plane two port $T^{(2)}$ by interchanging the two output terminals in each port of $T'$ as shown in figure 4.4.3. It can be easily shown that $T^{(2)}$ can then be written in terms of pseudopolynomials $A^{(2)}$, $B^{(2)}$, $C^{(2)}$ and the rational function $D^{(2)}$ in standard form as expressed in property 4.3.1', where

$$A^{(2)} = A', \quad B^{(2)} = C', \quad C^{(2)} = B', \; D^{(2)} = -D' \tag{4.4.8}$$

Step 3: A GGM section is then extracted from the two-port with chain matrix $T^{(2)}$ by iterating step 1 on $T^{(2)}$ to get a fully recursive half-plane lossless two-port chain matrix $T^{(3)}$. If pseudopolynomials $A^{(3)}$, $B^{(3)}$, $C^{(3)}$ and rational function $D^{(3)}$ represent $T^{(3)}$ in standard form as in property 4.3.1' then $B^{(3)}$ would have a factor $z_2$. Also, since from (4.4.4b) and (4.4.8) we have $C^{(3)} = C^{(2)}k_2{}^{(3)} = B'k_2{}^{(3)}$, where $k_2{}^{(3)}$ defines the GGM section presently extracted, and $B'$ has a factor $z_2$ we conclude that $C^{(3)}$ has a factor $z_2$. From this and the fact that $A^{(3)}\tilde{A}^{(3)} = B^{(3)}\tilde{B}^{(3)} + C^{(3)}\tilde{C}^{(3)}$ it follows that $A^{(3)}\bar{A}^{(3)} = 0$ for $z_2 = 0$ and for arbitrary $z_1$. Since $\bar{A}^{(3)} \neq 0$ for $|z_1| = 1$ and $|z_2| < 1$, we conclude $A^{(3)} = 0$ for $z_2 = 0$ and for arbitrary $z_1$. Consequently, $A^{(3)}$ has a factor $z_2$ and it is possible to write, for some pseudopolynomials $A^{(4)}$, $B^{(4)}$ and $C^{(4)}$ that

$$A^{(4)} = z_2A^{(3)}, \; B^{(4)} = z_2B^{(3)}, \; C^{(4)} = z_2C^{(3)} \tag{4.4.9}$$

Step 4: The last step in the synthesis cycle is to extract a $z_2$ type delay from $T^{(3)}$ as in figure 4.4.4 to produce a two-port with chain matrix $T^{(4)}$, which in standard form can be expressed in terms of $A^{(4)}$, $B^{(4)}$, $C^{(4)}$. Furthermore, $\bar{A}^{(4)} = \bar{A}^{(3)} \neq 0$ for $|z_1| = 1$, $|z_2| < 1$; $\bar{A}^{(4)} = \bar{A}^{(3)}$ can have at most finitely many zeros on $|z_1| = |z_2| = 1$. Also, it follows from (4.4.9) and losslessness of $T^{(3)}$ that

$$\tilde{A}^{(4)}A^{(4)} = \tilde{B}^{(4)}B^{(4)} + \tilde{C}^{(4)}C^{(4)} \tag{4.4.10}$$

and $\qquad \deg_2 B^{(4)} \leq \deg_2 A^{(4)}, \; \deg_2 C^{(4)} \leq \deg_2 A^{(4)}$

Thus, the two-port associated with $T^{(4)}$ is fully recursive half-plane lossless. Furthermore, note that

$$\deg_2 A^{(4)} = \deg_2 A^{(3)}-1 = \deg_2 A^{(2)}-1 = \deg_2 A'-1 = \deg_2 A-1,$$

where the first equality follows from (4.4.9); the second and the fourth from the fact that in step 1 we have $n_A = n_{A'}$; and the third from (4.4.8).

Consequently, after iterating $\deg_2 A$ times the cyclic algorithm described in Steps 1 through 4, we obtain a lossless chain matrix $T_f$ independent of $z_2$, which in standard form is described by $A_f = A_f(z_1)$, $B_f = B_f(z_1)$, $C_f = C_f(z_1)$ and $D = D_f(z_1)$.

<u>Terminal</u> <u>Step</u>: In the final step we extract another GGM section as in Step 1 to produce a fully recursive half-plane lossless two-port $S_0$ with $A_0$, $B_0$, $C_0$ and $D_0$ in standard form. Since $A_f$, $B_f$ are functions of $z_1$ only it follows from (4.4.5a) that $B_0 \equiv 0$. Also, since $\tilde{A}_0 A_0 = \tilde{B}_0 B_0 + \tilde{C}_0 C_0$ and $\bar{A}_0 = A_0 \neq 0$ for all $|z_1| = 1$ the 1-D transfer functions $(S_0)_{12} = D_0 C_0/A_0$ and $(S_0)_{21} = C_0/A_0$ are both well defined and of unit modulus on $|z_1| = 1$ i.e., they are all-pass functions.[2] The realization for such a two-port is shown in figure 4.4.5. The resulting composite filter structure is as shown in figure 4.4.6.

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[2]Note that $(S_0)_{12}$ and $(S_0)_{21}$ are not necessarily stable rational functions, i.e., may have poles in $|z_1|<1$.

## 4.5. DESIGN OF 2-D FULLY RECURSIVE HALF-PLANE FILTERS:

Two-dimensional filters with only two kinds of symmetries in their magnitude responses, namely the fan type symmetry and the circular symmetry are of practical interest. The locii of canstant gain in the $\omega_1$-$\omega_2$ plane for the fan filters are required to be approximate straight lines, whereas those for the circularly symmetric filters are required to be closed circles in an approximate sense. In addition, we also require the pass (or the stop) region of the fan filter to be the region approximately lying within the straight lines $\omega_1 = \alpha\omega_2$ and $\omega_1 = -\alpha\omega_2$ for some $0<\alpha<1$.

Our design proceeds by requiring the transfer function $S_{21} = C/\bar{A}$ (cf. equation (4.3.9a)) of the lossless two-port S to have the desired characteristics. However, unlike the corresponding problem in 1-D, due to nonfactorability of m-D polynomials it is in general not possible to find a pseudo-polynomial B satisfying (4.3.10) from A and C. To circumvent this problem it will be further assumed that the two-port is either symmetric i.e., $S_{11} = S_{22}$, $S_{21} = S_{22}$ or antimetric i.e., $S_{11} = -S_{22}$, $S_{21} = S_{12}$. Thus, in the symmetric and in the antimetric case we respectively have $B = -D\tilde{B}z_2^{n_A}$, $B = D\tilde{B}z_2^{n_A}$, whereas we also have $C = D\tilde{C}z_2^{n_A}$ in both cases. We next define two rational functions $S_1$ and $S_2$ as in (4.5.1) and 5.2) respectively for symmetric or antimetric two ports.

$$S_1 = (B + C)/\bar{A}, \quad S_2 = (B - C)/\bar{A} \qquad (4.5.1a,b)$$

$$S_1 = (B + jC)/\bar{A}, \quad S_2 = (B - jC)/\bar{A} \qquad (4.5.1'a,b)$$

From (4.5.1) it is easily verified that $S_1\tilde{S}_1 = S_2\tilde{S}_2 = 1$. Thus for each i, $|S_i| = 1$ for all $|z_1| = |z_2| = 1$ except possibly finitely many values where it is undefined. Furthermore, $\bar{A} \neq 0$ for $|z_1| = 1$, $|z_2| < 1$. Thus via Property 4.2.2a it follows that $S_i$, for each $i = 1,2$ in (4.5.1) must be a fully recursive half-plane all-pass function. Exactly same conclusions hold for $S_1$ and $S_2$ in (4.5.1'). Consequently, $S_1$, $S_2$ can be expressed as in (4.5.3), where $D_1$, $D_2$ and $A_1$, $A_2$ satisfy properties analogous to D and A in Property 4.2.3.

$$S_1 = -D_1A_1/\tilde{A}_1, \quad S_2 = -D_2A_2/\tilde{A}_2 \qquad (4.5.3a,b)$$

Note that even if A, B, C are real rational functions, $S_1$ and $S_2$ are real in (4.5.1ab) but not in (4.5.1'a,b). Thus, a symmetric filter can be realized by making use of the relation $S_{21} = C/\bar{A} = (S_1 - S_2)/2$, where the one-ports $S_1$ and $S_2$ are realized as in Appendix A. Although $S_{21} = C/\bar{A} = -j(S_1 - S_2)$ holds true in the antimetric case, a realization in terms of this last mentioned equation is not feasible due to the presence of the factor j unless complex filter realizations are called for. In this case, the pseudopolynomials A, B, C which are real, can be found from (4.5.1'a,b) and subsequently $S_{21}$ can be realized as being embedded in a real two-port S described by A, B, C in standard form. The design problem then boils down to appropriately choosing the real 1-D rational functions $D_1$, $D_2$, and real pseudopolynomials $A_1$, $A_2$ so that the frequency response requirements on $|S_{21}|$ are satisfied. This latter step may be carried out by using numerical optimization (e.g. Levenberg-Marquadt). For the purpose of numerical optimization, however, the following symmetry observations have the effect of reducing the number of parameters to be optimized.

Note that if $|S_{21}(\omega_1, \omega_2)|$ possesses either the fan-type or the circular-type symmetry then $|S_{21}(\omega_1, \omega_2)|$ must have, in particular, the so called quadrantal symmetry [14] i.e., $|S_{21}(\omega_1, \omega_2)|$ is unaltered if the signs of either $\omega_1$ or $\omega_2$ or both are changed. This requirement coupled with the stability property of $S_{21}$ demands that the denominater pseudopolynomial of $S_{21}$, and thus $A_1$ and $A_2$ in (4.5.3) satisfy a certain factorability property. It then proves to be expedient to carry out the optimization procedure on these factors rather than on the pseudopolynomials $A_1$, $A_2$.

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**APPENDIX:**

In this appendix we prove that a fully recursive symmetric half-plane all-pass function $H=H(z_1,z_2)$ can be synthesized as an interconnection of GGM sections (cf. Section 4.3) and $z_2$-type delays. This can be considered to be a generalized form of Schur's algorithm [9].

Let $k_1 = k_1(z_1) = H(z_1,0)$. Since H is as in Property 4.2.3 it follows from Property 4.2.2a that $|k_1|<1$ for all $|z_1|=1$ with the possible exception of finite number of values of $z_1$, where $|k_1|=1$. Thus, a $k_2$ satisfying (4.4.2) can be found i.e., $k_1$ and $k_2$ defines a GGM section. Consider next the function $H_1 = H_1(z_1,z_2)$ defined as in (4.A1.1), which can be interpreted as the residual transfer function after extraction of the GGM section just mentioned from $H_1$.

$$H' = (H - k_1)/(1 - \tilde{k}H) \qquad (4.A1.1)$$

From (4.2.17) it then follows that $H' = - pA'/B'$, where $pA' = DA + k_1\bar{A}$, $B' = \bar{A} + \tilde{k}_1DA$, p being the conjugate reciprocal factor of largest degree present in the numerator of $(DA + k_1\bar{A})$ when expressed in irreducible rational form. Next, since we have $\tilde{D} = D^{-1}$ it follows that $\tilde{p}\tilde{A}'/\tilde{A} = \tilde{D}(1-\tilde{k}_1H)$. Consequently, if $\deg_2 A' < \deg A$ then we would have $\tilde{k}_1H(z_1,0) =|H(z_1,0)|^2= 1$ for arbitrary $z_1$, which is impossible (cf. Property 4.2.2a). Thus, $\deg_2A'= \deg A$. It then clearly follows that $\tilde{p}\bar{A}' = \tilde{D}(\bar{A} + \tilde{k}_1DA) = \tilde{D}B'$, thus $H' = - D_1(A'/\bar{A}')$; $D_1 = \tilde{D}(\tilde{p}/p)$. Also, since $A' = - \bar{A}(H-k_1)$ it follows that $A' = 0$ for $z_2=0$ and arbitrary $z_1$. Thus, the pseudopolynomial $A'$ has a factor $z_2$. By defining $A_1$ via $zA_1 = A'$ we can write $H_1 = z_2H'$, where $H_1 = - D_1(A_1/\bar{A}_1)$. Note that $H_1$ can be constructed simply by extracting a $z_2$-type delay from $H'$. Clearly, $D_1$ satisfies condition (ii) of property 4.2.3. Also, by following arguments similar to that used after (4.4.7a,b) it can be shown that $\bar{A}_1 = \bar{A}' \neq 0$ for $|z_1| = 1$, $|z_2| < 1$ and the pseudopolynomials $A_1$ and $\bar{A}_1$ are coprime. Thus, condition (iii) and (iv) of Property 4.2.3 are satisfied by $H_1$, which has now been proved to be fully recursive symmetric half-plane lossless. Since $\deg_2A_1 = \deg_2A' - 1 = \deg_2A - 1$ the procedure just described when applied $\deg_2A$ times yields a circuit as shown in figure 4.A.1, in which the terminating section is an all-pass (not necessarily stable) in $z_1$ only.

# CHAPTER 5

# CONCLUSIONS AND FURTHER WORK

## 5.1 CONCLUSIONS:

Due to potential applications of multidimensional passive filtering schemes in manifold areas of signal processing e.g., including frequency filtering, modelling of random fields associated with detection and estimation of parameters of multidimensional signals, fundamental issues relating to the description of passive m-D systems and their synthesizability have been addressed in the present report. First, various classes of stable multidimensional polynomials essential to the discrete domain description of passive m-D systems has been delineated. The synthesizability of quarter plane type causal m-D digital filters has been investigated in a very general setting. Finally, due to its potential benefit to be derived from currently emerging parallel hardware architechtures [1], [2], a method of passive filtering within the framework of an alternative recursive scheme, namely, the fully recursive half-plane recursive scheme has been introduced and studied. Filter synthesis procedures within this framework has also been examined.

A major problem in processing such m-D discrete signals in real time is the large amount of data rate involved. Conventional digital filtering algorithms, which process data sequentially, is therefore, inappropriate for such processing purposes. In spite of the fact that, as in the 1-D case, the basic linear algebra operations such as vector matrix multiplication form the core of many m-D signal processing algorithms available, a detail study of their fast VLSI/optical implementation, which utilizes the underlying structure of multidimensional problems is yet unavailable. On the other hand, a consistent scattering formalism for passive multidimensional systems has begun to emerge as a result of the presently reported work, it should now be possible to undertake an investigation into the design of a broad variety of concurrent, numerically stable and fault tolerant m-D signal processing algorithms.

## 5.2. FURTHER WORK:

Study of novel multidimensional signal processing algorithms of the concurrent type for implementation in VLSI/optical architectures via the framework of passive scattering theory should thus form the major emphasis of future research in this area. Two generic considerations ensuing from the present work work may potentially form the basis of this investigation. To elaborate on the first, note that traditionally most m-D processing (filtering) have relied on the quarter-plane type recursion schemes in the causal order of data points. However, since unlike 1-D, in most multidimensional applications the independent coordinates describing the signal may not have temporal, but only spatial significance, to impose such a causality restriction is not only unnecessary, but it may cause severe drawback in fully exploiting the concurrency offered by the optical architechtures. Due to this, the fully recursive symmetric half-plane scheme, seems to be most appropriate for a large number of m-D signal processing problems of diverse nature. To justify this remark in 2-D, it may noted that the computational model under consideration have the remarkable property that all data points belonging to a row in the 2-D lattice space may be simultaneously (in parallel) computed from the data points belonging to an adjacent row in the 2-D lattice space. Furthermore, as has been demonstrated in the case of frequency filtering in chapter 4, the computation just referred to, in fact, involves computing the convolution of the adjacent rows with a fixed 1-D sequence determined by the transfer function of the filter. Since 1-D convolution, among other linear algebra operations, are known to easily yield to high speed optical architechtures using acousto-optic devices [1], the time complexity of the entire computation can be shown to be drastically less than filters using other recursive schemes. (In fact, the speed of this 2-D processing scheme proves to be of comparable order to that of a 1-D filter, when an implementation of this type is called for). Furthermore, since for certain 3-D applications, (e.g., in the processing of 3-D time varying imagery) this type of processing corresponds to processing all of the data points in a given frame of the image simultaneously, while the direction of filter recursion corresponds to the direction of flow of time, the processing scheme under consideration is not only adequate for fast optical implementation, but is the most natural choice unlike the somewhat contrived quarter plane causal recursive scheme, which has been almost universally

adopted in the m-D scattering based signal processing techniques so far.

A second possibility of optical implementation of passive filtering algorithms arises from the fact that 1-D digital lattice filters has been shown to be easily implementable via analog optical devices such as the single mode fiber and directional couplers [1]. The obeservation that the digital lattice section (also known as the Gray-Markel section) and variations of it can be implemented in terms of optics, can be potentially utilized in the implementation of passive m-D digital filters. This conclusion derives from the fact that, as shown in chapters 3 and 4 any passive m-D digital filter can, in fact, be implemented as interconnection of modular building blocks each of which can in turn be viewed as interconnections of a small number of digital lattice sections. To carry this point little further let us note that single mode optical fibers and directional couplers can be interpreted as passive digital two-port networks [1] and their propagation characteristics can, in fact, be described in terms of scattering parameters, which are exactly the tools in deriving a large variety of 1-D and m-D signal processing algorithms of interest to us in the present context. However, we are unaware of any existing work which develops this connection further so as to tailor specific m-D algorithms to fit into the optical architechtures.

References:

[1] H. John Caulfield, Samuel Horvitz, Gus P. Tricoles (ed.), Special issue on optical computing, Proc. IEEE, July 1984.

[2] S. Y. Kung, H. J. Whitehouse, T. Kailath, VLSI and modern signal processing, Prentice Hall, 1985.

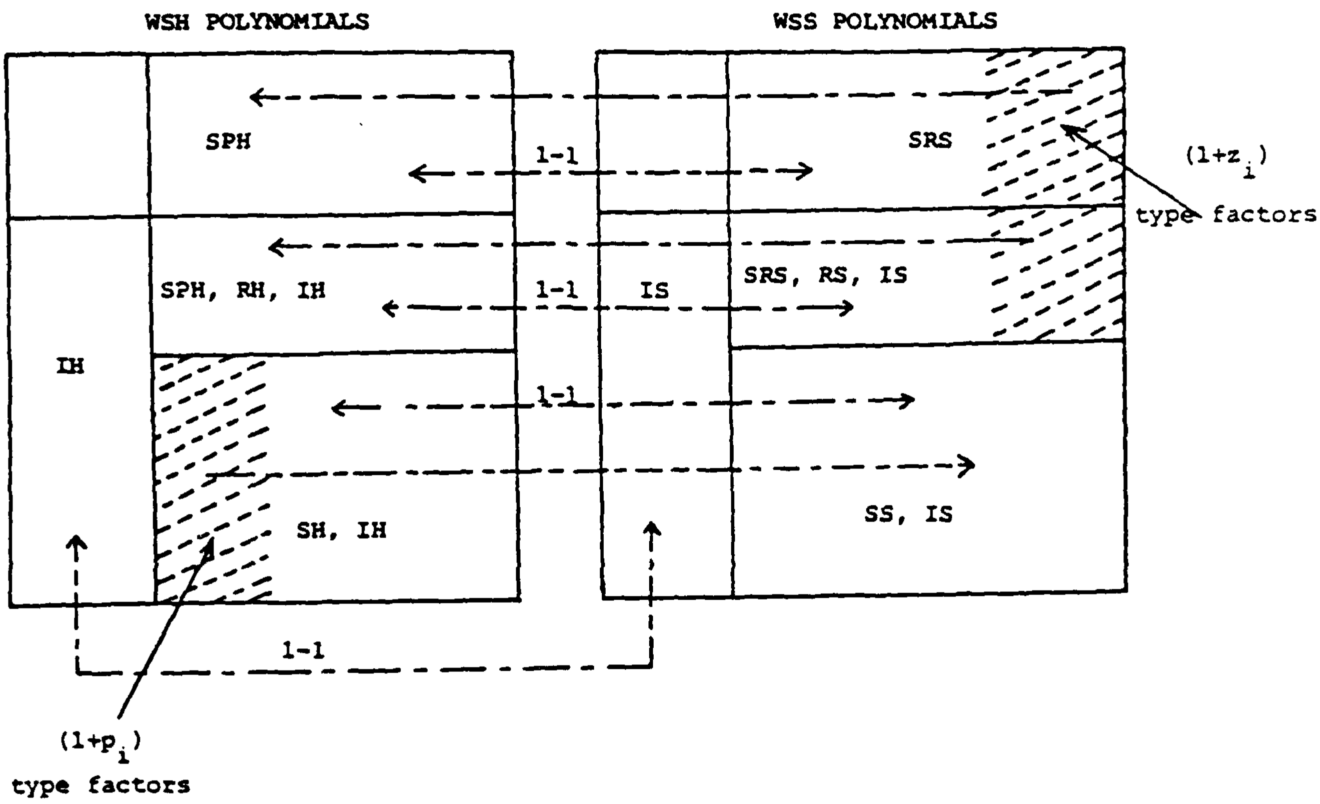
WSH POLYNOMIALS
WSS POLYNOMIALS
SPH
1-1
SRS
$(1+z_i)$
type factors
SPH, RH, IH
1-1
IS
SRS, RS, IS
IH
1-1
SH, IH
SS, IS
1-1
$(1+p_i)$
type factors

Figure 2.1

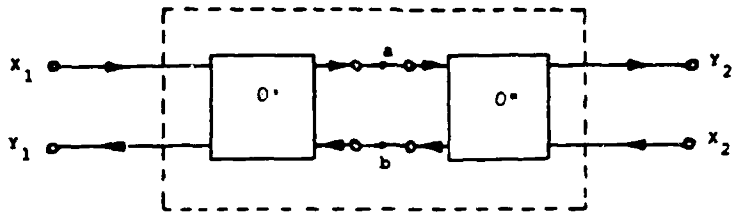


Figure 31. Network resulting from the factorization
$\Theta = \Theta'\Theta''$ where $[X_1 \ Y_1]^t = \Theta\,[Y_2 \ X_2]^t$ .

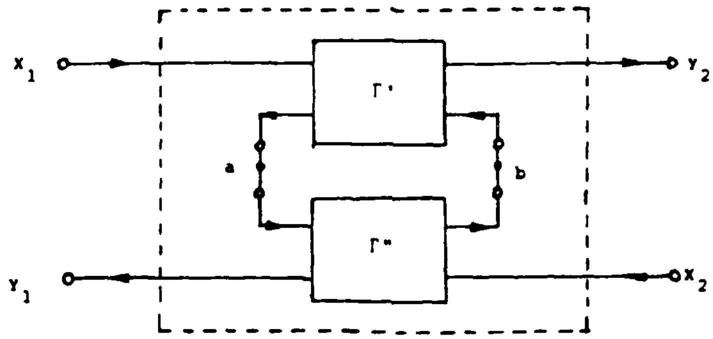


Figure 32. Network resulting from the factorization
$\Gamma = \Gamma'\Gamma''$ where $[X_1 \ Y_2]^t = \Gamma\,[Y_1 \ X_2]^t$

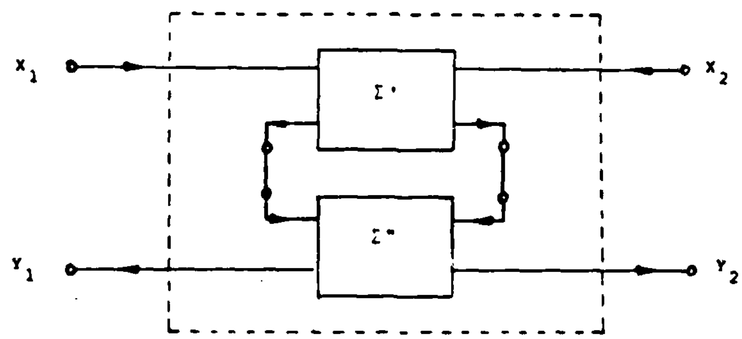


Figure 33. Network resulting from the factorization
$\Sigma = \Sigma'\Sigma''$ where $[Y_1 \ Y_2]^t = \Sigma\,[X_1 \ X_2]^t$

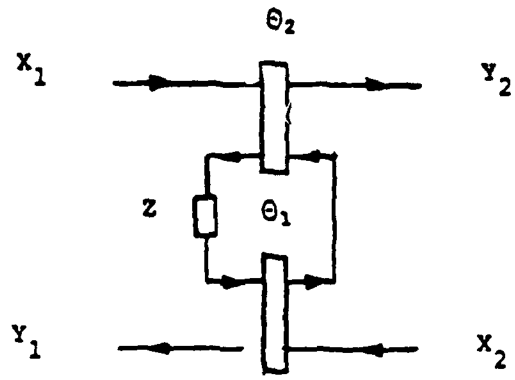


Figure 34. Elementary section with $n_1 = 1$ where rectangular boxes are Gray-Markel sections.

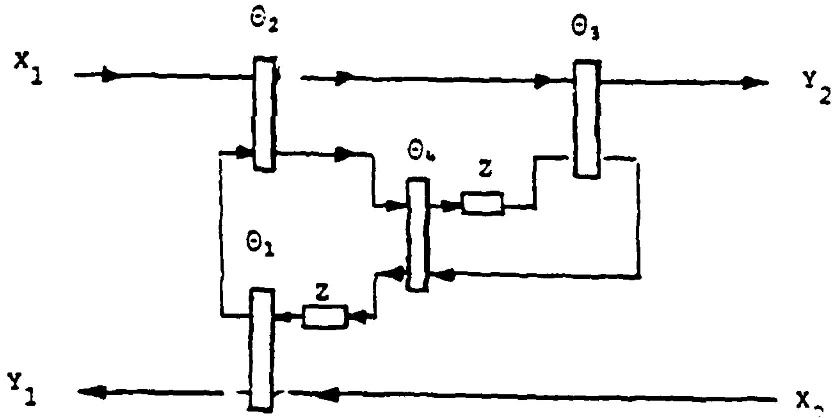


Figure 35. Elementary section with $n_1 = 2$.

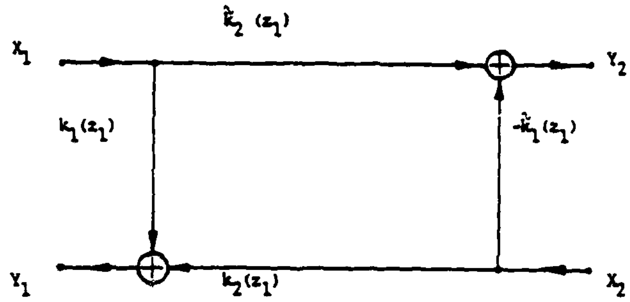


Fig. 4.4.1: A generalized Gray Markel section (G G M)

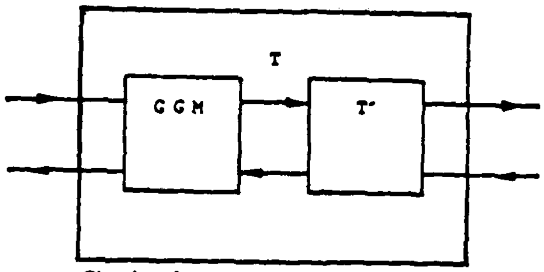


Fig. 4.4.2: Step 1; G G M extraction

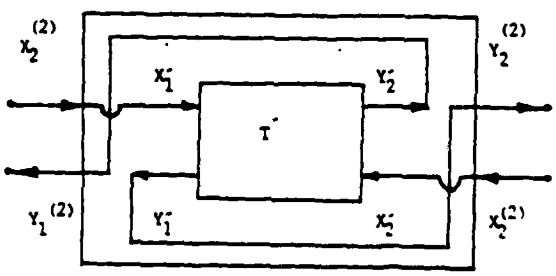


Fig. 4.4.3: Step 2; Two-port $T^{(2)}$ obtained from T'

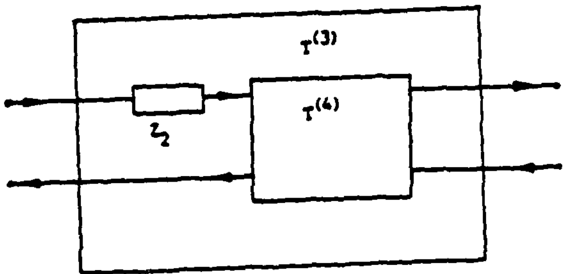


Fig. 4.4.4: Step 4: $Z_2$ - delay extraction

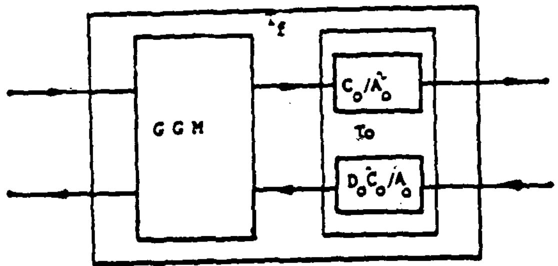


Fig. 4.4.5: Terminal Step in Synthesis

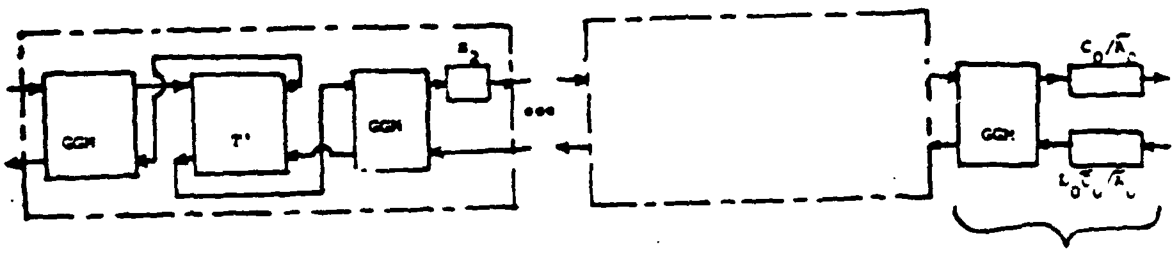


Fig. 4.4.6: Composite filter structure

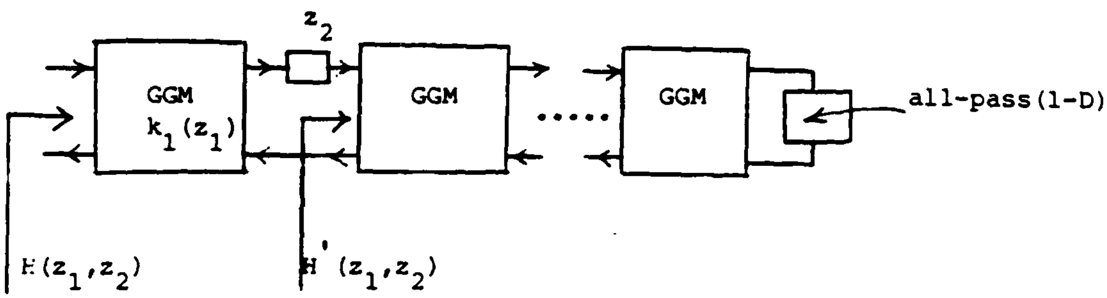


Figure 4.A.1:synthesis of a fully recursive symmetric all-pass one-port

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